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ALGEBRA

MADE EASY.

CHIEFLY INTENDED

FOR THE USE OF SCHOOLS.

BY

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PREFACE.

This work is intended to lead the pupil, by an easy transition, from the *principles* of arithmetic to those of algebra. I have endeavoured, throughout the work, to illustrate abstract symbols and operations by a reference to natural objects; and the arrangement, which has been adopted, is that which nature and expediency seem to suggest. The subjects do not follow each other in the order of *rules*, but according to the peculiar nature of the principles discussed, and the progressive difficulty of the operations. After the expositions of an important principle various problems are given, in which the principle is exemplified. By this means the inventive powers of the pupil are exercised, and he is led, at an early stage of his progress, to see the utility of the subject.

Instead of attempting to teach a great many rules of arithmetic — which are to many pupils useless, because their practical application is never learned — the simple and beautiful doctrine of equations should be taught; and if this be done by rational means it will elevate the intelligence of the pupil, and thus place him above the slavish observance of conventional forms and technical formulæ. By this process an entire system of algebra may not be taught: but because we cannot teach the whole of a science, it is no reason why we should not illustrate some of its simple and useful departments; — because we cannot teach a boy the method of finding the greatest common measure of two quantities, it is no reason why we should not explain the nature and use of equations; — or because standard writers on algebra happen to introduce the rule of surds at an early part of their works, will this be deemed a sufficient reason for keeping a boy ignorant of the nature of a recurring decimal?

An enlightened system of education ought to cultivate all

the susceptibilities of our nature: algebra, which is peculiarly called the analytic art, exercises in a way which no other subject can, those powers of analysis and abstraction, which would otherwise lie dormant and enfeebled. The philosopher, embracing within the comprehensive grasp of his intellect the known laws of the universe, was once a feeble-minded child; and in order that such a child may become a philosopher, its powers of analysis and induction should be cultivated, in a manner corresponding to its capabilities, at an early period of its intellectual development. While moral training, of a specific kind, should not be lost sight of, at the same time it is important to bear in mind, that the patient and reflective spirit which analysis engenders, gives a healthful tone to the character, and renders us more capable of appreciating and enjoying whatever is morally sublime or beautiful in nature or revelation.

We naturally acquire abstract ideas by a process of induction, — hence it is that a knowledge of the general symbols and operations of algebra is best communicated by an inductive method of instruction, whereby the mind of the pupil is led, by progressive steps, from the most simple particular cases, to those that are most complex and general. However, a judicious teacher will not fail, occasionally, to pursue a deductive process in elucidating certain abstract results, by showing the law of the formula in cases which come more within the range of our ordinary associations. In all algebraic investigations, each step should be thoroughly understood, before the next is attempted, or even presented to the eye of the pupil. By this means, thought is built upon thought — truth upon truth — until the pupil has, almost insensibly, acquired an accumulation of ideas. This plan has been appropriately called the constructive method of education, inasmuch as it is analogous to the way in which the most simple as well as the most gigantic of human contrivances are completed: — thus the ingenious mechanic lays stone upon stone, beam upon beam, until he has reared a vast and beautiful structure, exciting the wonder and admiration of the uninitiated observer. In this way too surprising results may be attained in education.

T. TATE.

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ALGEBRA MADE EASY.

DEFINITIONS AND NOTATION.

1. ALGEBRA teaches us a general method of computation, in which letters of the alphabet are used to represent numbers and quantities. Numbers that are unknown, but which may be found from certain things given in a question, are represented by x, or y, or any of the last letters of the alphabet; whereas numbers that are supposed to be known are represented by a, or b, or any of the first letters of the alphabet. When a figure is placed before any letter, it means, that the number for which the letter stands is to be multiplied by the figure. Thus $3x$ means, that the number for which x is put, is to be taken 3 times, that is, $3x=x+x+x$. In this example the 3, put before the x, is called the coefficient of x; and in the expression, $5x$, the coefficient is 5.

In like manner $a \times b$, or $a\,b$, means that the number for which a stands is to be multiplied by the number for which b stands; thus if $a=2$, and $b=4$, then $a\,b=2\times 4=8$.

The *expression*, $c\,d+e$, means that the number c is to be multiplied by the number d, and then this product is to be added to the number e; thus if c stands for, or is equal to 5, and $d=3$, and $e=7$, then in this case $c\,d+e=5\times 3+7=22$.

The *quantity* $\frac{x}{4}$ means that the number, for which x is

put, is to be divided by 4, or that we have to take the 4th part of x; thus if $x=8$, then in this case $\frac{x}{4}=\frac{8}{4}=2$.

When a quantity is multiplied by itself, there is a concise way of indicating the product. Thus, the 2nd power of a, or $a \times a$ is expressed by a^2, in the same way, we have, 6×6 expressed by 6^2; the 3rd power of a, or $a \times a \times a$ is expressed by a^3; in the same way, we have, $6 \times 6 \times 6$ expressed by 6^3, where the little 3 placed over the quantity shows the number of times that the quantity is to be multiplied by itself, — this number is called the *exponent*, or *index*. And so on to other powers.

When a bracket ( ) encloses two or more quantities, as $(a+b)c$, it means that a and b are first to be added together, and then multiplied by c; thus if $a=2$, $b=4$, and $c=3$, then $(a+b)c=(2+4) \times 3=6 \times 3=18$.

The various *signs* and *symbols*, used in arithmetic, are also employed in algebra. The meaning of other symbols, will be given in the course of the work.

In order to make the pupil more fully acquainted with the nature of algebraic symbols, he should calculate the following expressions: —

When $a=2$, $b=4$, and $c=6$.

1.	Calculate the value		of $a+2b-c$.	*Ans.* 4.
2.	...	...	of $a\,b+2c$.	*Ans.* 20.
3.	...	...	of $\frac{b}{2}+\frac{c}{3}$.	*Ans.* 4.
4.	...	...	of $(a+b)\,c$.	*Ans.* 36.
5.	...	...	of a^2+2b-c.	*Ans.* 6.

When $a=3$, $b=2$, $c=5$, and $n=2$.

6.	Calculate the value		of $(a-b)\,5+n$.	*Ans.* 7.
7.	...	...	of a^3-b^2+b+a.	*Ans.* 28.
8.	...	...	of $(b+c)^2+a$.	*Ans.* 52.
9.	...	...	of $\frac{a+c}{2}+b$.	*Ans.* 6.

PRINCIPLES AND OPERATIONS. — ADDITION, ETC.

2. We add quantities together when we collect them into one expression: thus we have the addition of $2x+3x=5x$; precisely in the same way as we have the addition of 2 *marbles* and 3 *marbles* equal to 5 *marbles; 2 fields* and 3 *fields* equal to 5 *fields;* or 2 of any thing whatever added to 3 of the same thing equal to 5 of that particular thing.

Again we have, $7x-4x=3x$; just in the same way as we have, 7 *oxen* less by 4 *oxen* equal to 3 *oxen;* or 7 *bags of nuts* less by 4 *bags of nuts* equal to 3 *bags of nuts.*

Let it be required to add together $4a$, $3a$, $5b$, and $4b$. Here collecting the a's together and the b's together, we have, $4a+3a+5b+4b=7a+9b$, precisely as we have 4 *cows* + 3 *cows* + 5 *sheep* + 4 *sheep* = 7 *cows* + 9 *sheep.*

If we add 4 to $x-4$ the sum will be x; because the quantity, $x-4$, just wants 4 to make it x. In order to illustrate this principle, let us suppose that John has *a bag of marbles,* containing any number you please, but that he is indebted 4 marbles to James, then the number of John's marbles will be, *the bag of marbles* − 4 *marbles.* But if I give him 4 *marbles,* or add 4 to his stock, he will then have *the bag of marbles,* for the 4 that I give him will just enable him to pay his debt without taking any out of his bag; hence it follows that $+4$ and -4 destroy each other, and similarly $+a$ and $-a$ destroy each other.

Again we have the addition of, $8b-5b+7a-3a=3b+4a$, where the 5 b's are taken away from the 8 b's, and the 3 a's from the 7 a's, just as we have,

8 *fields* − 5 *fields* + 7 *horses* − 3 *horses* = 3 *fields* + 4 *horses.*

Let us now take the sum of, $4b+3a-5a=4b-2a$; here there are more a's to be subtracted from the 4 b's than there are to be added, that is, there are 3 a's to be added, and 5 a's to be subtracted, therefore there will remain 2 a's to be subtracted from 4 b's.

The sum of $5x-4a-6a=5x-10a$; here we collect all the quantities to be subtracted from 5 x's, that is. since 4 a's are to be taken away, and 6 a's are also to be taken away, there must therefore be altogether 10 a's to be taken away; just in the same way, as we would put all the separate debts of a person together, in order to obtain the total debt, or money to be subtracted from his property.

In addition, therefore, we collect all the *like* quantities, or things of the same sort together, taking care to observe whether the quantities are to be added or subtracted.

The usual method of proceeding in addition is, to write the expressions under each other, and then collect the like terms together, as in the following example.

Ex. Add, $a-2b$, $2a+3b$, and $a+4b$ together.

Here collecting the b's together, we have, 5 b's for the sum, then collecting the a's together, we have 4 a's for the sum; so that the whole sum is $4a+5b$.

$$\begin{array}{r} a-2b \\ 2a+3b \\ a+4b \\ \hline 4a+5b \\ \hline \end{array}$$

EXAMPLES IN ADDITION AND SUBTRACTION.

1.	What is the sum of $5x+2b+6b$?	*Ans.* $5x+8b$.
2.	... ... of $2x+5x-x+6$?	*Ans.* $6x+6$.
3.	... ... of $9x-x+4-2$?	*Ans.* $8x+2$.
4.	... ... of $5x-3+2x-2$?	*Ans.* $7x-5$.
5.	... ... of $3x-2+a+2$?	*Ans.* $3x+a$.
6.	Add together $5x+2$ and $3x+7$.	*Ans.* $8x+9$.
7.	... $5a$ and $4x-5a$.	*Ans.* $4x$.
8.	... $3a-2$ and $5-a$.	*Ans.* $2a+3$.
9.	Subtract 5 from $3x+5$.	*Ans.* $3x$.
10.	... $3x$ from $7a+3x$.	*Ans.* $7a$.
11.	... $3a$ from $x+4a$.	*Ans.* $x+a$.
12.	... 8 from $2x+8$.	*Ans.* $2x$.
13.	... $5x$ from $5x+7$.	*Ans.* 7.

14. Subtract 4 from $2x-3$.		*Ans.* $2x-7$.
15. ...	$2x+2$ from $6x+2$.	*Ans.* $4x$.
16. ...	$5a+3$ from $7a+4$.	*Ans.* $2a+1$.
17. Add together $a+b$, and $a-b$.		*Ans.* $2a$.
18. ...	$2a-4$, $3a-2$, and $5-a$.	*Ans.* $4a-1$.

3. Let us now see what we do when we have to multiply a quantity by any number. If we have to multiply $4x$ by 3, we mean that $4x$ is to be repeated 3 times, that is, 3 times $4x=4x+4x+4x=12x$; just in the same way as 3 times 4 *marbles* are equal to 12 *marbles;* or 3 times 4 *fields* are equal to 12 *fields;* and so on.

The division of a quantity by any number is just the reverse of the operation of multiplication. Thus the 4th part of $12x$, or $12x\div4$, is equal to $3x$; precisely in the same way as the 4th part of 12 *oranges* is equal to 3 *oranges;* or the 4th part of 12 *bags of nuts* is equal to 3 *bags of nuts;* and so on.

Quantities may be multiplied in any order. Thus 4×3 is the same as 3×4. and 4 times x is the same as x times 4. In order to show this, let there be x marks or units, arranged in 4 horizontal rows, then taking the marks or units horizontally, the total number of units will be x units taken 4 times; and taking the marks or units vertically, the total number of units will be 4 units taken x times, therefore x units taken 4 times will be equal to 4 units taken x times, that is, $x\times4=4\times x$. To illustrate this important principle still further, let us find the cost of x lbs. of tea at 3*s.* per lb. Now if the tea were at 1*s.* per lb., it will be readily seen, that the cost would be x shillings; but as it is at 3*s.* per lb., the cost must be 3 times x shillings; but the cost of the tea, will also obviously be expressed by 3*s.* taken x times, or x times 3 shillings; therefore 3 times $x=x$ times 3.

1 1 1 1 . . . to x units.
1 1 1 1 . . . to x units.
1 1 1 1 . . . to x units.
1 1 1 1 . . . to x units.

Following out the same principle, let us now show, that

3 times $4a=4$ times $3a$. For this purpose, let there be 4 a's arranged in 3 horizontal rows, then taking the a's horizontally, the total number of a's will be 4 a's taken 3 times; and taking the a's vertically, the total number of a's will be 3 a's taken 4 times, therefore 4 a's taken 3 times is the same as 3 a's taken 4 times, that is, 3 times $4a=4$ times $3a$.

$$\begin{array}{l} a+a+a+a. \\ a+a+a+a. \\ a+a+a+a. \end{array}$$

EXAMPLES IN MULTIPLICATION AND DIVISION.

1. What are 5 times $3x$? *Ans.* $15x$.
2. ... 7 times $2a$? *Ans.* $14a$.
3. ... 4 times $5x$? *Ans.* $20x$.
4. Repeat $3x$ for 5 times. *Ans.* $15x$.
5. ... $2a$ for 7 times. *Ans.* $14a$.
6. ... $2ab$ for 4 times. *Ans.* $8ab$.
7. Show that 5 times x is the same as x times 5.
8. Add together 3 times $2x$, and 5 times $4x$. *Ans.* $26x$.
9. Add together 4 times $2x$, and 4 times $3x$. *Ans.* $20x$.
10. Add together 5 times $2a$, and 3 times $4b$. *Ans.* $10a+12b$.
11. Subtract 2 times $3x$ from 5 times $3x$. *Ans.* $9x$.
12. ... 4 times $2x$ from 4 times $4x$. *Ans.* $8x$.
13. What is the 4th part of $8x$? *Ans.* $2x$.
14. ... 5th part of $15x$? *Ans.* $3x$.
15. ... 3rd part of $12x$? *Ans.* $4x$.
16. If $6x$ be put into 3 equal parts, what will one of them be? *Ans.* $2x$.
17. Put 9 x into 3 equal lots of x's, and find how many there are in each. *Ans.* $3x$.

    Here, $9x=\overline{x+x+x}+\overline{x+x+x}+\overline{x+x+x}$.
18. Put $8x$ into 4 equal lots of x's, and find how many there are in each. *Ans.* $2x$.
19. What does the operation in the last example show? *Ans.* That the 4th part of $8x$ is $2x$.

20. How many times can $2x$ be taken out of $8x$?

Ans. 4 times; because $8x=2x+2x+2x+2x$.

21. What is the 5th part of $5x$? *Ans.* x.

22. To show, by a familiar illustration, that the number of times which 4 is contained in, or can be taken out of, $4x=x$ times.

1 1 1 1 1 to x times.	
1 1 1 1 1	ditto.
1 1 1 1 1	ditto.
1 1 1 1 1	ditto.

Here we have x units in each horizontal row; but as there are 4 horizontal rows there will be $4x$ units altogether. Now taking the marks vertically, we have x vertical columns, with 4 units in each, therefore, 4 units are contained in, or can be taken out of, the total number of units x times, that is, the number of times which 4 units can be taken out of $4x$ units $=x$ times.

SIMPLE EQUATIONS.

4. Equations enable us to solve difficult problems, more easily than they can be done by common arithmetic. An equation is formed when we have a certain quantity, or collection of quantities, equal to some other quantity or collection of quantities. Thus $5+4=2+7$ is an equation, or equality; and $2x=12$ is also an equation, where x must be such a number, that when it is taken 2 times it will produce 12. The number for which x here stands must obviously be 6, because $2\times 6=12$. Here it will be seen that the equation enables us to find what the x, or unknown number, must be.

The quantity to be found, or the unknown number, in any problem, is usually represented by x; and the *solution* of the resulting equation consists in finding the value of x, or the number that it must be, in order to make the quantities on the one side of the equation equal to those upon the other. The principle upon which the solution of an equation depends is this,—*whatever we do to one side of the equation, we must do the same thing to the other side, in order to keep up the equality.*

When there are a certain number of things given to find the value of one, we may do it by an equation; for example, if 3 lbs. of sugar cost 18*d.*, let it be required to find the cost of 1 lb. Here the price of one lb., multiplied by the number of lbs., must give the whole cost, that is,

The cost of 1*lb. in pence* $\times$ *no. lbs.* = *the whole cost in pence,* but the *no. lbs.*, in the example given, are 3, and *the whole cost* is 18*d.*,

$\therefore$ *The cost of* 1 *lb. in pence* $\times 3 = 18$ pence;

$\therefore$ The cost of 1 lb. in pence $= \frac{1}{3}$ of 18 pence $=6$ pence.

Now for the sake of conciseness let x be put for *the cost of* 1 *lb. in pence,* then 3 times x, or $3x$, must express the number of pence which 3 lbs. would cost; but by the question this is equal to 18 pence, hence we have,

$$3x=18 \text{ pence};$$

then as 18 pence are the cost of 3 things, one of them must cost the third of 18 pence, or 6 pence, that is,

$$x=\tfrac{1}{3} \text{ of } 18 \text{ pence}=6 \text{ pence}.$$

Here to find the value of x, we take the 3rd of each side of the equality: — Thus, the third of $3x$'s is x, and the third of 18*d.* is 6*d.*

In constructing, or forming an equation from any proposed problem, we in general find two distinct or separate expressions for the same thing, no matter what the thing may be, and then put these expressions equal to each other, as they must be if they are values of the same thing. In the preceding example, $3x$ are put equal to 18 pence, because they both express the same thing, that is, the whole cost.

As another illustration; let it be required to find how many lbs. of tea at 5*s.* per lb., must be given for 10 lbs. of coffee at 2*s.* per lb.

Here let x = the no. lbs. of tea.

We want now to find an expression for the cost of the tea. It will be readily seen, that x lbs. at 1*s.* each lb. will amount

to x shillings; and therefore x lbs. at 5*s.* each lb. will amount to 5 times x shillings, or $5x$ shillings.

$$\therefore\ 5x = \text{cost of the tea in shillings};$$

$$\text{but, } 2 \times 10 = \quad \ldots \quad \text{coffee} \quad \ldots$$

Now by the question, the cost of the tea is to be equal to the cost of the coffee,

$$\therefore\ 5x = 2 \times 10 = 20;$$

then taking the 5th part of each side of the equation,

$$x = 4, \text{ the no. lbs. required.}$$

PROBLEMS IN SIMPLE EQUATIONS.

1. A horse and a cow were bought for £24; now the horse cost 3 times as much as the cow. Required the value of each.

Here let us put the horse into cows. The horse will be equal to the value of 3 cows; hence we have,

Value of the cow + 3 times *value of the cow* = £24.

Collecting together *the values of the cow*, we have,

4 times *value of the cow* = £24.

Taking the 4th part of each side of the equality, we find,

Value of the cow = £6,

and as the value of the horse is 3 times the value of the cow, the value of the horse = 3 times £6 = £18.

Now it will not at all affect the accuracy of this process if we write x for *the value of the cow;* then

$$x = \text{the value of the cow in £'s},$$

$$\therefore\ 3x = \quad \ldots \quad \text{horse} \quad \ldots .$$

But by the question, the two together cost £24, that is,

the value of the cow + the value of the horse = £24,

$$\therefore\ x + 3x = 24;$$

collecting the x's, we have,

$$4x = 24;$$

taking the 4th part of each side of the equality,

$$x = 6, \text{ value of the cow in £'s;}$$

and the value of the horse = 3 times £6 = £18.

2. A man's age is 4 times that of his son's age, and their united ages amount to 50 years. What is the son's age?

$$\text{Let the son's age} = x,$$
$$\therefore \text{ the father's age} = 4 \text{ times } x = 4x.$$

But by the question, their ages added together = 50 years, that is,

$$\text{the son's age} + \text{the father's age} = 50 \text{ years},$$
$$\therefore\ x + 4x = 50 \text{ years},$$

collecting the x's, we have,

$$5x = 50 \text{ years},$$

taking the 5th part of each side of the equation, we have,

$$x = 10 \text{ years}.$$

We may readily show that this result is correct; for 10 + 4 times 10 = 50, which answers to the conditions of the question.

In order to show the construction and solution of the foregoing equation, $x + 4x = 50$, the following questions may be put by the Teacher to the Pupil.

Teacher. Why is the father's age represented by $4x$?

Pupil. Because the question tells us that his age is 4 times that of the son's.

Teacher. Why do we put x for the *son's* age?

Pupil. In order to get an expression for the father's age, in keeping with the language of the question.

Teacher. (Writing x upon the black board.) What have I written down?

Pupil. The son's age.

Teacher. (Now writing down $+4x =$.) What have I now written down?

Pupil. The father's age to be added to the son's.

Teacher. What must I put this sum equal to?

Pupil. You must put the sum equal to 50 years.

Teacher. Why?

Pupil. Because the question tells us, that the two ages taken together make up 50 years.

Teacher. What will this equation enable us to find?

Pupil. The number x must be.

Teacher. (Writing the equation, $5x = 50$ years.) How do we get this equation?

Pupil. By putting the x's together.

Teacher. What must I now do in order to get the value of x?

Pupil. Take the fifth part of each side of the equality.

A similar analysis may be made of any of the problems hereafter given. The teacher must constantly bear in mind, that in order for his instruction to be effective it must always have the character of *progressive development.*

3. Divide 24 marbles between John and Andrew, so that John may receive 3 times as many as Andrew.

Let $x =$ Andrew's share,

then as John has to receive 3 times this number,

$3x =$ John's share;

but the two shares taken together, are equal to 24; that is,

Andrew's share + John's share = 24,

$$\therefore x + 3x = 24;$$

putting the x's together, we have,

$$4x = 24;$$

taking the 4th of each side of the equation, we have,

$x = 6$, the no. Andrew will receive;

$\therefore$ John will receive 3 times 6, or 18.

4. If 5 be added to 4, it will give 3 times a certain number. Required the number.

$$\text{Let } x = \text{the number,}$$
$$\text{then } 3x = 3 \text{ times the number;}$$

but by the question,

$$3 \text{ times no.} = 5+4,$$
$$\therefore\ 3x = 5+4,$$
$$\therefore\ 3x = 9;\ \text{and } x = 3, \text{ the number.}$$

5. A butcher bought a sheep and a cow for 84*s.*; now he paid 6 times as much for the cow as he paid for the sheep. What did he pay for each?

$$\text{Let } x = \text{no. shillings paid for the sheep,}$$

but as he paid 6 times as much for the cow,

$$6x = \text{no. shillings paid for the cow.}$$

Now the cost of the two together amounts to 84*s.*, that is,

$$\text{cost sheep} + \text{cost cow} = 84s.,$$
$$\therefore\ x+6x = 84s.$$

collecting the x's,

$$7x = 84s.,$$

taking the 7th part of each side of the equation,

$$x = 12s., \text{ the price of the sheep,}$$
$$\therefore \text{ the price of the cow} = 6 \text{ times } 12s. = 72s.$$

We may readily prove this result, because 12*s.* for the sheep added to 6 times 12*s.* for the cow, will just produce 84*s.*

6. A gentleman left £20 to be divided between his two servants: but to the first he left 3 times as much as to the second. How much should each receive?

Ans. £15. and £5.

7. A farmer bought a certain number of sheep for 72*s.*, at 12*s.* a head. How many sheep did he buy?

$$\text{Let } x = \text{the number bought;}$$

then as each sheep cost 12*s.*,

$$12 \times x = \text{no. shillings which the whole cost.}$$

But we are told that the whole cost 72*s.*,

$$\therefore\ 12x = 72;$$

then dividing each side of the equation by 12, we have,

$$x = 6, \text{ the number of sheep.}$$

8. If I had 3 times the money I have in my pocket, I should have 15*d*. How much have I? *Ans.* 5*d*.

9. A certain number taken 9 times is equal to 6. What is the number? *Ans.* $\frac{2}{3}$.

10. A boy being asked how many marbles he had, said, if I had twice as many more, I should have 21. How many had he? *Ans.* 7.

Let $x =$ no. marbles,

then $2x =$ twice as many. But by the question,

No. + twice No. $= 21$,

$\therefore\ x+2x=21.$

11. The area of a rectangle is 18 sq. ft., and the length is 6 ft. Required the breadth.

Let $x =$ breadth in ft.; then as the area of a rectangle is found by multiplying the length by the breadth,

$6\times x =$ area rectangle,

$\therefore\ 6\times x=18$; and $\therefore\ x=3$ ft.

12. How many yards of calico at 4*d*. per yard, must be given for 2 yards of cloth, at 8*s*. per yard?

Let $x =$ no. yards of calico,

then $4\times x =$ cost calico in pence;

and $2\times 96 =$ cost cloth in pence.

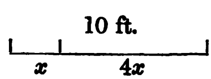
But by the question the cost of the one must be equal to the cost of the other.

$\therefore\ 4\times x=2\times 96$; and $\therefore\ x=48$, the no. yards.

13. Divide a rod 10 ft. long into two parts, so that the one part may be 4 times the length of the other.

Let x=the less part,

then $4x$=the greater part.



Now these two parts are to make up 10 ft.,

$\therefore\ x+4x=10$ ft.,

collecting the x's,

$5x=10$ ft.,

taking the 5th part of each side of the equation,

$x=2$ ft. the less,

and the greater $=4$ times 2 ft. $=8$ ft.

14. If 3 times a certain number be taken from 7 times the same number, the remainder will be 8. What is the number?

Let $x =$ the number,

$\therefore$ $3x =$ three times the number,

and $7x =$ seven times the number,

$$\therefore\ 7x - 3x = 8,$$

taking the $3x$ away from the $7x$, we have,

$$4x = 8,$$

dividing each side of the equation by 4,

$x = 2$, the number.

15. Divide £27 among three persons, A, B, and C, so that B may have twice as much as A, and C three times as much as B.

Let A's share $= x$,

then B's share $= 2x$.

But C's share is to be 3 times B's, that is,

C's share $= 3$ times $2x = 6x$.

Now the sum of these three shares is to make up £27.

$$\therefore\ x + 2x + 6x = £27,$$

collecting the x's,

$$9x = £27,$$

$\therefore$ $x =$ £3, A's share.

Therefore B's $= 2$ times £3 $=$ £6,

and C's $= 3$ times £6 $=$ £18.

Here it will be observed that we put the x for the least share.

16. The sum of £50 is to be divided among 2 men, 3 women, and 4 boys, so that each man shall have twice as much as each woman, and each woman three times as much as each boy. Required the share of each.

Let each boy's share $= x$,

then each woman's share $= 3x$,

and each man's share $= 2$ times $3x = 6x$.

Hence we have,

$$\text{The share of the 4 boys} = 4x,$$
$$\text{the share of the 3 women} = 3 \text{ times } 3x = 9x,$$
$$\text{and the share of the 2 men} = 2 \text{ times } 6x = 12x.$$

But the sum of all these shares is to amount to £50,

$$\therefore\ 4x+9x+12x = £50,$$

collecting the x's together,

$$25x = £50; \text{ and } \therefore\ x = £2.$$

As each boy's share is £2; each woman's = 3 times £2 = £6; and each man's = 2 times £6 = £12.

17. Divide £21 among three persons A, B, and C, so that B may have four times as much as A, and C one half as much as B.

Let A's share $= x$,
then B's ... $= 4x$,
and C's ... $= \frac{1}{2}$ of B's $= \frac{1}{2}$ of $4x = 2x$,

But the sum of all these shares is to make up £21,

$$\therefore\ x+4x+2x = £21.$$

Solving this equation we find A's share = £3, B's = £12, and C's = £6.

18. I want to divide 35*d.* among 3 poor persons, so that the second may have twice as much as the first, and the third twice as much as the second. How much must I give to each? *Ans.* 5*d.*, 10*d.*, and 20*d.*

19. Divide 20 into three parts, such that the second shall be 5 times the first, and the third equal to the difference between the first and second. *Ans.* 2, 10, and 8.

20. A grocer mixes tea at 5*s.* per lb. with an equal weight of tea at 3*s.* per lb. How many lbs. must there be of each kind, so that the whole may sell for 32*s.*?

Let x = no. lbs.,
then $5x$ = price of the 1st kind in shillings,
and $3x$ = ... 2nd ... ...

But the sum of these two prices is to amount to 32*s.*,

$$\therefore\ 5x+3x=32,$$

collecting the x's,

$$8x=32,$$

taking the 8th part of each side of the equation,

$$x=4, \text{ the no. lbs.}$$

21. A farmer sold two equal lots of sheep for £10.; the price of each sheep in one lot was 12*s.*, and in the other lot 13*s.* How many sheep were there in each lot? *Ans.* 8.

22. How many lbs. of sugar at 8*d.* per lb. must be given for 4 lbs. of tea at 6*s.* per lb.? *Ans.* 36.

23. If a certain number be multiplied by 5, the product will be 5 more than 35. What is the number? *Ans.* 8.

24. John has 5 times the number of marbles that Thomas has; but the difference of their numbers is 12. How many has each? *Ans.* 3, and 15.

25. A farmer bought an equal number of sheep, cows, and horses, for £18.; for each sheep he paid £1.; for each cow £3.; and for each horse £5. Find the number of each. *Ans.* 2.

Examples in Numerical Equations.

	Equations.	Answers.
1.	$7x=21$	$x=3$
2.	$5x+x=24$	$x=4$
3.	$x+2x+3x=12$	$x=2$
4.	$7x=40+16$	$x=8$
5.	$8x-3x=15$	$x=3$
6.	$7x+2x-4x=25$	$x=5$
7.	$9x-2x=24-3$	$x=3$

PROBLEMS IN SIMPLE EQUATIONS.

5. In the preceding examples, all the unknown quantities are upon one side of the equation, and all the known ones

are upon the other; but in the following class of equations, this will not be found to be the case. The first operation we have now to perform, is to bring all the unknown quantities to one side of the equation, and all the known ones to the other. The principle upon which this operation depends, is, that *we may increase or decrease equal things by the same quantity, without destroying the equality:* thus, if two boys have an equal number of marbles, and if I give 2 marbles to one, I must also give 2 marbles to the other, in order that they may still have the same number; or if I take 2 marbles away from the one, I must also take 2 marbles away from the other, in order that they may still have the same number.

1. If you were to give me 4*d.*, I should have 10*d.* How much have I?

Let x=my no. of pence; then when you give me 4*d.*, the number of pence I should then have will be $x+4$; but this number, we are told, in the question, is 10, therefore we have the equation,

$$x+4=10.$$

Taking 4 away from the left side of the equation, and 4 at the same time away from the right side, to keep up the equality, we have,

$$x=6, \text{ the no. of pence.}$$

2. I took 3 marbles from a boy and then he had 5 left. How many had he at first?

Let x=the no. at first,
then $x-3$=the no. left.

But by the question, this number left is equal to 5.

$$\therefore\ x-3=5,$$

adding 3 to each side,

$$x=8, \text{ the number at first.}$$

3. A farmer bought a cow and a horse for £13; now the horse cost £3 more than the cow. What did each cost?

Let x=cost of the cow in pounds.

But as the horse cost £3 more than the cow,

$$x+3=\text{cost of the horse in pounds.}$$

Now the two together amount to £13,

$$\therefore\ x+x+3=13\,;$$

collecting the x's, and taking 3 from each side,

$$2x=10\,;$$

taking the half of each side,

$$x=5,\text{ the cost of the cow in pounds,}$$
$$\text{and the cost of the horse}=£5+£3=£8.$$

4. Divide 12 into two parts, so that the one part may exceed the other part by 2. *Ans.* 5 and 7.

5. Three times a certain number, is equal to the number itself increased by 8. What is the number?

$$\text{Let } x=\text{the number,}$$
$$\text{then } 3x=\text{three times the number,}$$
$$\text{and } x+8=\text{the number increased by 8.}$$

Therefore by the equality stated in the question,

$$3x=x+8,$$

taking x from each side,

$$2x=8,$$

dividing each side by 2, we have,

$$x=4,\text{ the number.}$$

6. At an election 864 men voted. Now the successful candidate had a majority of 78. How many voted for each? *Ans.* 393 and 471.

7. A man walked a certain distance; now if he had gone 8 miles further he would have travelled 3 times the distance which he did. What distance did he travel? *Ans.* 4 miles.

8. A person bought some rice for 4*d.*, and then got 3*lbs.* of sugar; now the whole amounted to 25*d.* How much a lb. did he pay for the sugar? *Ans.* 7*d.*

9. A farmer by selling 7 sheep receives 24*s.* more than when he sells 5 sheep. What is the price of each? *Ans.* 12*s.*

10. A grocer sold 8 lbs. of tea for 24*s.*, and thereby lost twice as much as 1 lb. cost him. How much did 1 lb. cost?

Let $x =$ cost price of 1 lb., in shillings,

then $8x =$ ... 8 lbs., ...

$\therefore$ *Loss in shillings* $= 8x - 24$.

But by the question, we have also,

Loss in shillings $= 2x$.

Now these values for the same thing must be equal,

$$\therefore\ 8x - 24 = 2x,$$

adding 24 to each side,

$$8x = 2x + 24,$$

taking $2x$ away from each side,

$$6x = 24$$

$\therefore\ x = 4$, shillings per lb.

We may easily prove, or verify, this result: thus, the cost price of the tea will be 8 times 4*s.* or 32*s.*; then the loss will be 32*s.* $-$ 24*s.* $=$ 8*s.*, which it will be observed is just twice 4*s.*, or the cost price of 1*lb.*

11. A man bought a mule, a cow, and a horse for £21; he paid £2 more for the cow than he paid for the mule, and £5 more for the horse than the cow. What did he pay for the mule?

Let $x =$ price of the mule in £'s.

then $x+2 =$ ... cow ...

and $x+2+5 =$ ... horse ...

Now these prices taken altogether make up £21.

$$\therefore\ x + \overline{x+2} + \overline{x+2+5} = 21.$$

Collecting the x's, and also the numbers together, we have,

$$3x + 9 = 21.$$

Taking 9 from each side,

$$3x = 12.$$

$\therefore\ x =$ 4*l.*, the price of the mule.

12. Divide £35 among three persons, A, B, and C, so that B, shall have £8 more than A, and C £4 more than B.

Ans. A's $=$ £5, B's $=$ £13, and C's $=$ £17.

13. Tom has got $4d.$ more than James, and Matthew has got $2d.$ more than Tom. Now, the whole of their money amounts to $19d.$ How much has each? *Ans.* $3d.$, $7d.$, and $9d.$

14. I want to divide some nuts among a certain number of boys. Now, if I give 4 nuts to each boy, I shall have 2 nuts to spare; and if I give 3 to each boy, I shall have 8 to spare. How many boys are there?

Let x = the number of boys.

In solving this question we shall find two separate values for *the number of nuts* in terms of x, and then, by putting these values equal to each other, we shall obtain an equation for finding the unknown number.

Now, if 1 nut be given to each boy, there will be x nuts given away; and if 4 nuts be given to each, there will be 4 times x nuts given away.

$\therefore 4x =$ number of nuts distributed at first.

But the total number is 2 more than this.

$\therefore$ *Total no. nuts* $= 4x + 2$.

Again, $3x =$ no. nuts distributed in the second case.

But the total number is eight more than this.

$\therefore$ *Total no. nuts* $= 3x + 8$.

$$\therefore 4x + 2 = 3x + 8.$$

Taking $3x$ from each side,

$$x + 2 = 8.$$

Taking 2 from each side,

$x = 6$, the no. of boys.

15. An author has a certain number of books to sell; if he sell them at $3s.$ each, he will lose £5.; but if he sell them at $4s.$ each he will gain £15. Required the number of copies.

Let $x =$ no. copies, then

From the first part of the question, we find,

Cost price in shillings $= 3x + 100$.

But from the second part of the question, we also find,

Cost price in shillings $= 4x - 300$.

$$\therefore 4x - 300 = 3x + 100.$$

Adding 300 to each side,

$$4x = 3x + 400.$$

Taking $3x$ away from each side,

$$x = 400, \text{ the no. of copies.}$$

16. A man, by selling a certain number of oranges at 3*d.* each, gained 10*d.*; but if he had sold them at 2*d.* each, he would have gained 1*d.* How many did he sell? *Ans.* 9.

Here it will be necessary to obtain two separate values for the total cost of the oranges.

17. Being sent to purchase a certain number of oranges, I found that I should just lay out my money when the oranges are 3*d.* each; but if they were 2*d.* each, I should have 7*d.* left. How many had I to purchase? *Ans.* 7.

18. Two travellers set out at the same time from two towns, forty-two miles apart, with the intention of meeting. One of them walks three miles an hour, and the other four miles. In what time will they meet?

Let x = the no. of hours required.

Then, as the distance passed over is equal to the product of the number of hours by the distance travelled in each hour, we have,

$3x$ = no. miles travelled by the one,

and $4x$ = ... by the other.

But, as they meet, they must together have gone over the whole distance,

$$\text{consequently, } 3x + 4x = 42$$
$$7x = 42$$
$$\text{and } x = 6, \text{ no. hours.}$$

19. A farmer bought a certain number of sheep at 9*s.* each, and as many more at 13*s.* each, and sold them all at 12*s.* each, and thereby gained 16*s.* How many did he purchase? *Ans.* 16.

6. EXAMPLES IN NUMERICAL EQUATIONS.

1. Let $7x + 3 = 21 - 2x$, find x.

Adding $2x$ to each side,

$$9x + 3 = 21.$$

Subtracting 3 from each side,

$$9x = 18.$$

Dividing each side by 9,

$$x = 2.$$

2.	$2x+4=12$	*Ans.* $x=4$
3.	$3x+2=18+x$	$x=8$
4.	$7x-4=31.$	$x=5$
5.	$4x-3=x+9$	$x=4$
6.	$8x-4=3x+26$	$x=6$
7.	$5x-8=10-x$	$x=3$
8.	$2x-x=14-6x$	$x=2$

7. RULE OF TRANSPOSITION.

We transpose a quantity when we take it from one side of an equation to the other; but in doing so we must **always** change the sign of the quantity we transpose.

In order to prove this rule, let us take the equation,

$$8x-4=8+4x.$$

Adding 4 to each side,

$$8x=8+4x+4.$$

Now by this operation, the 4 has passed from the left side to the right; but in doing so its sign has been changed from $-$ to $+$.

Subtracting $4x$ from each side of the last equation, we have,

$$8x-4x=8+4.$$

Now by this operation, the $4x$ has passed from the right side to the left; but in doing so its sign has been changed from $+$ to $-$. Hence the rule of transposition.

8. *We may change the signs of all the quantities in an equation.* In order to show this, let us take the equation,

$$8-3x=-2.$$

Transposing all the quantities, we have,

$$2=3x-8,$$

or what is obviously the same,

$$3x-8=2.$$

Here then we observe that *all* the signs of the quantities, in the original equation, are changed.

PRINCIPLES AND OPERATIONS.—MULTIPLICATION.

9. A *bracket* ( ) in algebra is used to connect, or throw into one distinct expression, any number of quantities. In multiplying $(2x+4a)$ by 3, or in finding the product $(2x+4a)\times 3$, we have not merely to take $4a$ three times, but also $2x$ three times, that is, we have to take the whole quantity within the bracket three times, thus,—

$$\begin{array}{c} 2x+4a \\ 2x+4a \\ 2x+4a \\ \hline \end{array}$$

$$3 \text{ times } (2x+4a) = 3 \text{ times } 2x + 3 \text{ times } 4a = 6x+12a.$$

Precisely in the same way as we would do if we had to increase a man's property, consisting of 2 *fields* + 4 *horses*, three times; for we should not only increase the *fields* 3 times, but we should also increase the *horses* 3 times, in order to increase the whole property 3 times; thus,—

3 times (2 fields + 4 horses) = 6 fields + 12 horses.

Examples.

1. Repeat $4x+3$ five times.	*Ans.*	$20x+15$
2. ... $3x+2$ four times.	...	$12x+8$
3. Multiply $(x+6)$ by 7.	...	$7x+42$
4. What is the product of $(5x+1)$ by 3?	...	$15x+3$
5. Find the product of $(2x+3)\times 2$.	...	$4x+6$
6. ... ... of $(5x+2a)\times 3$.	...	$15x+6a$
7. Multiply $(x+2a+3)$ by 2.	...	$2x+4a+6$
8. ... $(2x+a+1)$ by 4.	...	$8x+4a+4$

10. PROBLEMS IN SIMPLE EQUATIONS.

1. If 4 be added to a certain number, and the sum be multiplied by 5, the product will be equal to the number added to 32.

Let $x =$ the no.,

then $x + 4 =$ the no. added to 4.

and $(x + 4)\ 5 =$ the sums multiplied by 5.

But this product, by the question is equal to $x + 32$, the *number* added to 32.

$$\therefore (x + 4)\ 5 = x + 32,$$

Performing the multiplication,

$$5\ x + 20 = x + 32,$$

Taking x from each side,

$$4\ x + 20 = 32,$$

Taking 20 from each side,

$$4x = 12.$$

These two last operations might have been performed at once by the rule of transposition. See Art. 7. Thus transposing the x and 20, we have,

$$5x - x = 32 - 20,$$

$$\therefore 4x = 12,$$

Dividing each side by 4,

$x = 3$, the number.

2. A person in trade, at the end of the first year added £14 to his capital; and at the end of the second year he found that he had doubled his last year's capital, and then he had £78 more than his original capital. Required his original capital.

Let $x =$ the original capital,

then $x + 14 =$ capital at the end of the 1st year,

$\therefore (x + 14)\ 2 =$ capital at the end of the 2d year,

But by the question this must be equal to $78 + x$,

$$\therefore (x + 14)\ 2 = 78 + x,$$

Performing the multiplication,

$$2x + 28 = 78 + x,$$

$\therefore x =$ £50.

3. John has 3 times the number of marbles that Thomas has; but if 6 be given to each, John will then have only twice the number. How many had each? *Ans.* 18 and 6.

4. A and B commence trade with the same capital. At

the end of one year A gains £5, and B £65, and then B's capital is three times that of A's. Required the original capital.

Let $x =$ the original capital,

then $x+5 =$ A's capital at the end of the 1st year,

and $x+65 =$ B's capital at the end of the 1st year,

But B's capital is now 3 times A's,

$$\therefore x+65 = 3(x+5),$$

$$\therefore x+65 = 3x+15,$$

Subtracting x, and then 15 from each side, we have,

$$50 = 2x; \text{ and } \therefore x = £25.$$

5. Two boys, A and B, have the same number of marbles; but if I give 8 to A, and only 1 to B, then A will have twice the number that B has. How many had each at first? *Ans.* 6.

6. The difference of two numbers is 3, and if the less be taken from 4 times the greater, the remainder will be 33. Required the numbers. *Ans.* 7 and 10.

Let $x =$ the less,

then as the greater must be 3 more than the less,

$x+3 =$ the greater,

$$\therefore 4(x+3)-x = 33,$$

$$\therefore 4x+12-x = 33,$$

$$\therefore 3x = 21; \text{ and } x = 7.$$

7. The difference of two numbers is 2, and if twice the less be taken from 3 times the greater, the remainder will be 9. What are the numbers? *Ans.* 5 and 3.

8. How many gallons of spirits at 8*s.* per gallon, must be mixed with 5 gallons at 12*s.*, so that the mixture may sell for 9*s.* per gallon?

Let $x =$ no. gals. at 8*s.*,

then $8x =$ price of ditto in shillings,

and the price of the second kind of spirits = 5 times 12*s.*

= 60 shillings,

$\therefore 8x+60 =$ *total price of the mixture in shillings.*

Again, $x+5=$ no. gals. in the whole mixture.

But this mixture is to be at 9*s*. per gallon,

$\therefore$ $9(x+5) =$ *total price of the mixture in shillings.*

Here we have two distinct expressions for the same thing

$$\therefore 9(x+5) = 8x+60,$$

Performing the multiplication, $9x+45 = 8x+60$,

$$\therefore x = 15, \text{ no. gals.}$$

9. How much tea at 6*s*. per lb., must I mix with 10 lbs. at 3*s*., so that the mixture may sell for 4*s*. per lb.?

Ans. 5 lbs.

10. How much water must be mixed with 5 gals. of spirits at 12*s*. per gal., so that the mixture may sell for 8*s*. per gal.?

Let $x =$ no. gals. of water,

then $x+5 =$ ... in the whole mixture,

$\therefore$ $8(x+5) =$ *cost of the mixture* in shillings.

But the *cost of the mixture* $= 5 \times 12s. = 60$ shillings.

$$\therefore 8(x+5) = 60,$$

and $8x+40 = 60$,

$$\therefore x = 2\tfrac{1}{2} \text{ gals.}$$

11. How many gallons of spirits at 24*s*. per gal., must be mixed with 6 gals. of water, so that the mixture may be worth 8*s*. per gal.?

Ans. 3 gallons.

13. A grocer bought 32 lbs. of tea, and by selling it so as to gain 2*s*. per lb., he got £8 for the whole. What did he pay per lb.?

Let $x =$ cost price in shillings per lb.,

then $x+2 =$ selling ... ...

$\therefore$ $32(x+2) =$ no. of shillings received for the whole.

But the sum received for the whole is 160 shillings,

$$\therefore 32(x+2) = 160.$$

Dividing each side by 32,

$$x+2 = 5; \ \therefore x = 3 \text{ shillings.}$$

14. If 6 be added to a certain number, and the sum multiplied by 7, the product will be 63. Required the number.

Ans. 3.

15. A person gave some money among 5 poor persons; if he had given $3d.$ apiece more than he did, he would have given away $25d.$ What did he give to each? *Ans.* $2d.$

16. A rectangle is 6 ft. long, and if it were 2 ft. broader, its area would be 30 sq. ft. Required the breadth. *Ans.* 3 *ft.*

17. The difference of two numbers is 3, and 3 times the less is equal to 2 times the greater. Required the numbers.

Ans. 6 and 9.

18. A company upon settling their reckoning, find that they have $3s.$ each to pay; but observe, that if there had been 4 more, they should have had only $2s.$ each to pay. How many persons were there?

Let $x =$ the no. persons.

Now, since each person has to pay $3s.$,

$$3x = \textit{the reckoning in shillings.}$$

Again, $x+4 =$ the no. of persons increased by 4.

But with this number they have to pay $2s.$ each,

$$\therefore 2\,(x+4) = \textit{the reckoning in shillings},$$

$$\therefore 3x = 2(x+4);$$

Performing the multiplication,

$$3x = 2x + 8;$$

Taking $2x$ from each side,

$x = 8$, the no. persons.

19. A certain number of lbs. of sugar cost $6s.$; now, if there had been 3 lbs. more for the same money, each lb. would have cost $6d.$ Required the number of lbs. *Ans.* 9 lbs.

20. It took a certain number of feet of matting 3 ft. wide to cover a floor, but it took 60 ft. in length more, when the width was 2 ft. Required the length of the first kind of matting.

Let $x =$ the length of the matting,

then, $3x =$ *surface of the floor;* but we have also,

$$2(x+60) = \textit{surface of the floor},$$

$$\therefore 3x = 2\,(x+60),$$

$\therefore x =$ 120 ft. of matting.

21. It took a certain number of stones, 4 ft. long and 3 ft.

wide, to pave a yard ; but it took 80 more stones when the width of each was 2 ft. Required the number of stones of the first size. *Ans.* 160.

22. Divide £60 among A, B, and C, so that B may have £2 more than A, and C four times as much as A and B together. *Ans.* £5, £7. and £48.

EXAMPLES IN NUMERICAL EQUATIONS.

1.	$7(x+1) = 3(4+2x)$	*Ans.* $x = 5$
2.	$5x-1 = 2(1+x)+3$	... $x = 2$
3.	$7(1+2x)+2 = 3(4x+2)+9$	... $x = 3$
4.	$4x+2 = 2(x+6)-4$	... $x = 3$
5.	$5(x+3) = 21-x$	... $x = 1$
6.	$3(x-1)-x = 9$	... $x = 6$

PRINCIPLES AND OPERATIONS. — MULTIPLICATION.

11. Hitherto we have not had a minus quantity to multiply by another quantity; let us therefore now see what we do when we have to multiply, for example, $(a-x)$ by 3. As the entire quantity within the bracket is to be repeated 3 times, we have,

$$3 \text{ times } (a-x) = \overline{a-x} + \overline{a-x} + \overline{a-x} = 3a-3x.$$

And in like manner,

$$2 \text{ times } (3x-4a) = \overline{3x-4a} + \overline{3x-4a} = 6x-8a.$$

where we observe that the *signs* of the quantities in the *product* are the same as the corresponding quantities *within* the bracket. In order to illustrate this principle, let us suppose that there are 3 persons, having the same amount of *property*, and the same amount of *debt*, then each man's wealth will be equal to *property* − *debt;* and the total amount of their wealth will be,

3 times (*property* − *debt*) = 3 times *property* − 3 times *debt.*

Because, it will be readily seen, that while the *property* is increased 3 times, there will also be 3 times the *debt* to be taken away from the property so increased.

Again, let us suppose that there are x marks, or units, written down in 3 horizontal rows,

11,111 . . . to x units
11,111 . . . to x units
11,111 . . . to x units

and that we cut off two marks from each, then the number of marks remaining in each row will be $x-2$, and the number in the 3 rows will be,

$$3 \text{ times } (x-2) = 3x-6;$$

because, if there had not been any cut off, the total number would have been $3x$; but as there have been 2 cut off from each row, the total number cut off will be 3 times 2, or 6.

12. The expression $ax+bx = (a+b)x$, just in the same way as $2x+3x = (2+3)x$; because the x is first taken a times, and then b times; therefore the total number of times which x is taken will be $(a+b)$ times.

EXAMPLES IN MULTIPLICATION.

1.	Take $(2x-3)$ five times.	*Ans.* $10x-15$.
2.	Multiply $(3-4x)$ by 4.	... $12-16x$.
3.	... $(4x-2a)$ by 2.	... $8x-4a$.
4.	... $(x-4)$ by 9.	... $9x-36$.
5.	Find the product of $3(2a-3x)$.	... $6a-9x$.
6.	... ... $7(5x+a-1)$.	... $35x+7a-7$.
7.	... ... $2(2x-1)+3(x-2)$.	... $7x-8$.

13. PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. Ralph and Charles have the same number of marbles. Ralph loses 2, and Charles loses 5, and then Ralph has double the number that Charles has. How many had each boy at first?

Let $x =$ the no. at first,
then $x-2 =$ no. that Ralph has after losing 2,
and $x-5 =$ ... Charles ... 5.

But the question tells us that the former is double the latter.

$$\therefore\ x-2 = 2(x-5).$$

Performing the multiplication,

$$x-2 = 2x-10,$$

Adding 10 to each side, and then subtracting x,

$$8 = x, \text{ or } x = 8, \text{ the number each had.}$$

2. A farmer sold 8 sheep, and then the number he had at first was 3 times the number remaining. How many had he at first? *Ans.* 12.

3. Divide 28 into two parts, such that 3 times the greater shall be equal to 4 times the less.

$$\text{Let } x = \text{the less,}$$

then, as the two numbers make up 28,

$$28-x = \text{the greater,}$$

$$\therefore 4x = 3(28-x), \text{ then solving this equation,}$$

$$x = 12, \text{ the less,}$$

$$\text{And the greater} = 28-12 = 16.$$

This result may readily be verified, since we have, 3 times 16 = 4 times 12.

4. Divide 16 into two parts, such that 3 times the less, shall be equal to the greater. *Ans.* 4 and 12.

5. The united ages of a man and his son make up fifty years; but twice the father's age is equal to eight times the son's. Required their respective ages. *Ans.* 40 and 10.

6. A gentleman gave 22*s.* among eight persons; to a certain number he gave 2*s.* each, and to the rest 4*s.* each. How many were there of each class?

Let x = no. persons in the 1st class,
then $8-x$ = ... 2d ...
$\therefore 2x$ = no. shillings given to the 1st class,
and $4(8-x)$ = ... 2d ...

But the sum of these two quantities is equal to twenty-two shillings,

$$\therefore 2x+4(8-x) = 22.$$

Performing the multiplication,

$$2x+32-4x = 22,$$

Collecting the x's,

$$32-2x=22.$$

Adding $2x$ to each side, so as to have the sign of the x positive, or $+$,*

$$32=22+2x.$$

Subtracting 22 from each side, and then dividing by 2,

$$5=x, \text{ or } x=5.$$

Hence the number in the 2nd class $=8-5=3$.

7. A butcher bought eleven sheep, in two lots, for £5. 8*s.*; for the first lot he paid 9*s.* a head, and for the second 12*s.* How many were there in each? *Ans.* 8 and 3.

8. A person bought eight pounds of tea for 41*s.*, one part at 7*s.* per lb., and the other part at 4*s.* How many lbs. of each did he purchase? *Ans.* 3 and 5 lbs.

9. Tea at 5*s.* per lb. is mixed with tea at 3*s.*; now 12 lbs. of the mixture are sold for £2. 12*s.* How much was there of each sort? *Ans.* 8 and 4 lbs.

10. Two labourers, A and B, earn the same wages; A receives 2*s.* per day, and B 3*s.*; now the sum of the days that they were at work is 10. How many days was each man employed? *Ans.* 6 and 4.

11. A farmer bought 8 sheep; now, if he had paid 2*s.* a head less than he did, the money paid would have been £4. Required the price of each. *Ans.* 12*s.*

EXAMPLES IN NUMERICAL EQUATIONS.

1. Find the value of x in the equation,

$$3(x-2)+2(x+4)=4(5-x).$$

Performing the multiplication,

$$3x-6+2x+8=20-4x,$$

$$\text{Collecting, } 5x+2=20-4x,$$

By transposition, or adding $4x$ to each side, and subtracting 2,

$$9x=18, \text{ and } x=2.$$

* This may also be done by changing the signs of all the quantities. See Art. 8.

2. Find the value of x in the equation,

$$4(5x-2)+2x+3(2x+1) = 107.$$

Performing the multiplication,

$$20x-8+2x+6x+3 = 107,$$

Collecting, $28x-5 = 107,$

$$\therefore x = 4.$$

3. $3(x-4)+2x = 13$ *Ans.* $x = 5.$
4. $4(2x+2)+3(x-1) = 27$ ... $x = 2.$
5. $5(x-3)+2 = 2(x+1)$ ... $x = 5.$
6. $2(4x-3)-3x = 9.$ ... $x = 3.$

PRINCIPLES AND OPERATIONS. — FRACTIONS.

14. Fractions in algebra have the same meaning that they have in arithmetic. Thus the fraction $\frac{x}{2}$, means that $\frac{1}{2}$ is to be repeated for x times; and $\frac{a}{n}$, that the fractional unit $\frac{1}{n}$ is to be repeated a times.

But a fraction also means, that the numerator is to be divided by the denominator. In order to show this extension to the meaning of a fraction; let us take x oranges, and cut each orange into three equal parts, then $\frac{1}{3}$ of the orange taken x times will evidently be the same as $\frac{1}{3}$ of x oranges, that is x times $\frac{1}{3} = \frac{1}{3}$ of x. Thus, the $\frac{1}{2}$ *of a field* taken 3 times, is the same as $\frac{1}{2}$ *of three fields.* Hence we may view the fraction $\frac{x}{3}$ in two ways: 1st. It is the third part of unity taken x times. 2d. It is the third part of x units, or x divided by 3.

15. A fraction multiplied by the denominator gives the numerator: thus, 3 times $\frac{x}{3} = \frac{x}{3} + \frac{x}{3} + \frac{x}{3} = x$; just in the same way as $\frac{1}{3}$ of *a field,* taken 3 times = *the whole field;* or $\frac{1}{3}$ of *any thing whatever* taken three times = *the whole of that particular thing.*

Again, the half of *a thing,* must be taken 2 times to make *the whole thing;* the fifth of *a thing* must be taken 5 times

to make *the whole thing;* and the n^{th} part of *any thing* must be taken n times to make *the whole thing,* whatever it may be.

16. As the numerator of a fraction tells us the number of times that the fractional unit is taken, it follows that we take a fraction 2 times by taking the numerator 2 times; that we take a fraction 3 times by taking the numerator 3 times; and so on. Thus,

$$2 \text{ times } \frac{x}{5} = \frac{2x}{5}, \text{ and conversely } \frac{2x}{5} \div 2 = \frac{x}{5};$$

$$3 \text{ times } \frac{x}{5} = \frac{3x}{5}, \text{ and conversely } \frac{3x}{5} \div 3 = \frac{x}{5}; \text{ and so on.}$$

17. A fraction may always be cleared of its denominator, by multiplying by that denominator, or some multiple of it. Thus,

$$6 \text{ times } \frac{x}{2} = \overline{\frac{x}{2} + \frac{x}{2}} + \overline{\frac{x}{2} + \frac{x}{2}} + \overline{\frac{x}{2} + \frac{x}{2}} = 3x;$$

because every two of the halves make a whole.

Thus we also have; the third of *a house* taken 12 times $=$ 4 *houses,* because every three of the thirds make a whole; the fifth of *a field* taken 15 times $=$ 3 *fields,* because every five of the fifths make a whole; and the half of *any thing* taken 8 times $=$ 4 *of that thing,* because every two of the halves make a whole.

In all these cases the number of wholes is found by dividing the multiplier by the denominator. Thus, 15 times $\frac{x}{5} = 3x$, where 15 divided by 5 gives 3. In like manner $\frac{x}{7} \times 28 = 4x$.

These principles will be found very useful in solving fractional equations.

18. PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. Divide 12 into two parts, so that the one shall be the third of the other.

Let $x =$ the one part, then

$$\frac{x}{3} = \text{the other part.}$$

But the sum of these parts is equal to 12,

$$\therefore\ x + \frac{x}{3} = 12.$$

In order to get rid of the fraction, multiply each side by 3, then,

$$3x + x = 36,$$

$$\therefore x = 9, \text{ the one part;}$$

$$\text{and the other} = 12 - 9 = 3.$$

2. Divide a rod of 20 inches long, into two parts, so that the one may be one-fourth of the other.

Ans. 4 and 16 inches.

3. My purse and money make together 16*s.*, but my money is $\frac{1}{3}$ of the value of my purse. What is the value of my purse? *Ans.* 12*s.*

4. A tradesman sold goods for 22*s.*, and thereby gained one-tenth of the cost-price. Required the cost price.

Let $x =$ cost price in shillings,

$$\therefore\ \frac{x}{10} = \text{gain in shillings.}$$

But, cost + gain = selling price,

$$\therefore\ \ x + \frac{x}{10} = 22.$$

Multiplying each side by 10 to get rid of the fraction,

$$10x + x = 220,$$

or, $11x = 220$; and $x = 20$ shillings.

5. A horse and gig cost £20; now the gig was worth $\frac{1}{3}$ of the horse. Required the value of the horse. *Ans.* £15.

6. The half of a number added to 3, is equal to one third of the number added to 7. Required the number.

Let $x =$ the number, then we have the equation,

$$\frac{x}{2} + 3 = \frac{x}{3} + 7.$$

Here in order to clear this equation of fractions, we must

multiply each side by 6, the product of the denominators 2 and 3. See Art. 17.

$$\therefore 3x+18 = 2x+42,$$

$$\therefore x = 24, \text{ the number.}$$

7. Divide £63 among three persons, A, B, and C, so that A shall have $\frac{1}{3}$ of B's share, and C shall have $\frac{1}{6}$ of B's. *Ans.* 42, 14, and £7.

8. What number is that, whose fourth part exceeds its fifth part by 1? *Ans.* 20.

9. After spending 8*d.*, I had $\frac{1}{5}$ of my money remaining. How much had I at first? *Ans.* 10*d.*

10. A butcher by selling a sheep for 15*s.*, gained one-half of what the sheep cost. How much did it cost? *Ans.* 10*s.*

11. If 4 be added to one-third of a certain number, the sum will be 9. Required the number. *Ans.* 15.

12. If $\frac{1}{2}$ of a certain number be diminished by 2, the remainder will be $\frac{1}{3}$ of the number. Required the number. *Ans.* 12.

13. Find a number such that $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of it added together shall amount to 36.

Let x = the number, then we have the equation,

$$\frac{x}{3}+\frac{x}{4}+\frac{x}{6} = 36.$$

Here we might clear the equation of fractions by multiplying each side by the product of all the denominators, as we did in example 6.; but, in the present case, it will be more convenient to multiply each side by 12, which is the least common multiple of the numbers 3, 4, and 6. Multiplying every term, therefore, by 12, we have,

$$4x+3x+2x = 432,$$

Collecting the x's,

$$9x = 432, \text{ and } x = 48.$$

14. The $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a number added together, exceed the number itself by 2. Required the number. *Ans.* 24.

15. A post is one-tenth in the water, one-fourth in the ground, and 13 feet above the water. Required the length of the post. *Ans.* 20 feet.

16. After paying away one-fifth of my money, I had 6*d.* more than the half of my money left. How much had I at first?

Let $x =$ the no. pence,

then $\frac{x}{5} =$ money paid away,

$\therefore x - \frac{x}{5} =$ money left.

But, by the question, this is equal to $6 + \frac{x}{2}$

$$\therefore x - \frac{x}{5} = 6 + \frac{x}{2},$$

Multiplying each side by 10, the common multiple of 2 and 5,

$$10x - 2x = 60 + 5x,$$
$$\therefore x = 20 \text{ pence}.$$

17. A man after spending one-fourth of his income, found that he had £10 more than the eighth of his income left. Required his income. *Ans.* £16.

18. If you take away one-seventh of my money, I shall have 2*d.* more than 10*d.* How much have I? *Ans.* 14*d.*

19. Find two numbers whose sum is 14, and difference one-third the less.

The solution of a problem will frequently be rendered more simple and elegant, by departing from the usual method of supposing x to represent the quantity to be determined. In solving the present problem we shall derive an equation entirely free from fractions, by supposing,

$$3x = \text{the less},$$

then by the question,

$$\text{the difference} = \tfrac{1}{3} \text{ of } 3x = x.$$

Now the greater = the less + the difference,

$$\therefore \text{the greater} = 3x + x = 4x.$$

But the numbers make up 14,

$\therefore\ 3x + 4x = 14$, and $\therefore\ x = 2$,

$\therefore$ the less $= 3x = 6$; and the greater $= 14 - 6 = 8$.

20. My son's age is, at present, one-fourth of mine; but 4 years after this, his age will be one-third of mine. What are our ages?

Let $4x =$ the father's age,

$\therefore\ x =$ the son's age.

But after 4 years the father's age will be $4x + 4$, and then the son's age will be $x + 4$. Now by the question the latter quantity is $\frac{1}{3}$ the former,

$$\therefore\ \tfrac{1}{3}(4x + 4) = x + 4.$$

Multiplying by 3, to get rid of the fraction, we have,

$$4x + 4 = 3x + 12,$$

$$\therefore\ x = 8, \text{ the son's age.}$$

And the father's age $= 4$ times $8 = 32$ years.

21. The tail of a fish weighed 3lbs., the body twice as much as the head and tail together, and the head as much as the tail and one-fifth of the body. What was the weight of the fish?

Here, although the whole weight of the fish is the thing required in the question, yet it will be most convenient to suppose,

the weight of the head $= x$, in lbs.

then the weight of the head and tail $= x + 3$,

and the weight of the body $= 2$ times $(x + 3) = 2x + 6$.

But the weight of the head $=$ the weight of the tail $+$ one-fifth of the weight of the body.

$$\therefore\ x = 3 + \frac{1}{5}(2x + 6).$$

Multiplying every term by 5,

$$5x = 15 + 2x + 6,$$

$\therefore\ x = 7$ lbs. the weight of the head.

Then weight of the body $= 2$ times $(7 + 3) = 20$lbs.

Therefore weight of the whole fish $= 7 + 20 + 3 = 30$lbs.

22. John's age is one-fifth of mine, and Tom's is one-half

of John's. Now our united ages amount to 39 years. Required my age. *Ans.* 30 years.

Here to avoid fractions, $10x$ must be put for my age.

23. One-third the number of trees in a garden are apple trees, one-fourth pear trees, and there are 35 plum trees. How many are there altogether? *Ans.* 84.

19. EXAMPLES IN NUMERICAL EQUATIONS.

1. Solve the following equation:

$$\frac{x}{2}+\frac{x}{4}=20+\frac{x}{8}.$$

Here we shall get rid of all the fractions by multiplying every term by 8,

$$\therefore\ 4x+2x=160+x.$$

Collecting, and subtracting x from each side,

$$5x=160, \text{ and } \therefore\ x=32.$$

2. Given the following equation to find x,

$$\frac{4x-6}{3}-1=\frac{12-2x}{6}.$$

Multiplying each side by 6, we have,

$$8x-12-6=12-2x,$$

$$\therefore\ x=3.$$

3. $\frac{x}{3}+\frac{x+2}{4}=4.$ *Ans.* $x=6$.

4. $\frac{x}{4}+\frac{x}{5}=\frac{x}{8}+13.$ ... $x=40$.

5. $\frac{x+1}{5}=\frac{x+2}{6}+1.$ ... $x=34$.

6. $\frac{x}{2}+\frac{x}{4}+x=\frac{x}{3}+17.$ ... $x=12$.

7. $x-\frac{x}{2}+8=20.$ ... $x=24$.

8. $2x-\frac{x}{3}-4=28-x.$ ... $x=12$.

9. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{x}{4}+5$ ... $x=15$.

PRINCIPLES AND OPERATIONS.—FRACTIONS.

20. We take the half of a fraction, when we diminish the fractional unit two times; thus the half of $\frac{x}{3} = \frac{x}{6}$, because the fractional unit $\frac{1}{6}$ is two times less than $\frac{1}{3}$; thus the 6th of *a field* will be the half of the 3rd of *a field*. We take the third of a fraction, when we diminish the fractional unit three times; thus the third of $\frac{x}{4} = \frac{x}{12}$, because the fractional unit $\frac{1}{12}$ is three times less than $\frac{1}{4}$; thus the 12th of *a loaf* will be the 3rd of the 4th of *a loaf*. And so on. Generally the nth part of $\frac{x}{a} = \frac{x}{na}$, because $\frac{1}{na}$ is n times less than $\frac{1}{a}$. Moreover, 2 times the 3rd of $\frac{x}{5} = 2$ times $\frac{x}{15} = \frac{2x}{15}$, that is $\frac{2}{3}$ of $\frac{x}{5} = \frac{2x}{15}$; or generally m times the nth of $\frac{x}{a} = m$ times $\frac{x}{na} = \frac{mx}{na}$, that is, $\frac{m}{n}$ of $\frac{x}{a} = \frac{mx}{na}$, where we multiply the numerators together for the new numerator, and the denominators for the new denominator.

21. If we increase, or decrease, the denominator of a fraction any number of times, we must also increase, or decrease, the numerator the same number of times, in order to keep the value of the fraction the same. Thus $\frac{x}{3} = \frac{4x}{12}$, because out of each third we can cut four twelfths, so therefore there must be 4 times the number of twelfths that there are thirds; thus the 3rd of *a yard* will be the same as the 12th of 4 *yards*. And generally $\frac{x}{a} = \frac{mx}{ma}$, because the fractional unit $\frac{1}{ma}$ is m times less than $\frac{1}{a}$, and therefore the former must be taken m times more than the latter, in order to have the same value.

Or thus,

$$\text{By Art. 15., } a = \frac{na}{n}.$$

Decreasing each side of the equality b times, we have,

$$\frac{a}{b} = \frac{na}{nb}.$$

In order to add or subtract fractions, it is necessary that they should be expressed in the same fractional part of unity, or that they should have the same denominator.

$$\text{Thus } \frac{x}{3} + \frac{x}{4} = \frac{4x}{12} + \frac{3x}{12} = \frac{7x}{12};$$

$$\text{and } \frac{x}{4} - \frac{x}{5} = \frac{5x}{20} - \frac{4x}{20} = \frac{x}{20}; \text{ or generally,}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad + cb}{db}.$$

22. Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

First we have, $\frac{a}{b} \div c = \frac{a}{cb}$.

But this result is d times too little, for our divisor here is d times too much.

$$\therefore \frac{a}{b} \div \frac{c}{d} = d \text{ times } \frac{a}{cb} = \frac{ad}{cb}.$$

Here it will be observed that we invert the divisor and proceed as in multiplication.

23. EXAMPLES IN FRACTIONS.

1. What is $\frac{1}{2}$ of $\frac{3x}{4}$? *Ans.* $\frac{3x}{8}$.

2. ... $\frac{3}{4}$ of $\frac{x}{2}$? ... $\frac{3x}{8}$.

3. ... $\frac{2}{3}$ of $\frac{4x}{7}$? ... $\frac{8x}{21}$.

4. Bring $\frac{2x}{3}$ to ninths. ... $\frac{6x}{9}$.

5. ... $\frac{5a}{2}$ to eighths. ... $\frac{20a}{8}$.

6. Bring $\frac{a}{2}$ and $\frac{3x}{4}$ to twelfths. *Ans.* $\frac{6a}{12}$ and $\frac{9x}{12}$.

7. ... $\frac{33x}{9}$ to its least terms.

Here dividing numerator and denominator by 3,

$$\frac{33x}{9}=\frac{11x}{3}, \textit{ Ans.}$$

8. Bring $\frac{12x}{8}$ to its least terms. *Ans.* $\frac{3x}{2}$.

9. ... $\frac{15x}{12}$ ... ... $\frac{5x}{4}$.

10. Add $\frac{2x}{3}$ and $\frac{x}{6}$ together. ... $\frac{5x}{6}$.

11. ... $\frac{x}{2}$ and $\frac{2x}{5}$ ... ... $\frac{9x}{10}$.

12. ... $\frac{x}{3}$ and $\frac{2x}{9}$ ... ... $\frac{5x}{9}$.

13. ... $\frac{a}{3}$ and $\frac{1}{2}$. ...

Here multiplying numerator and denominator of the first by 2, and the second by 3, we have,

$$\frac{a}{3}+\frac{1}{2}=\frac{2a}{6}+\frac{3}{6}=\frac{2a+3}{6}, \textit{ Ans.}$$

14. Add $\frac{x}{3}$, $\frac{x}{4}$, and $\frac{x}{2}$ together. *Ans.* $\frac{13x}{12}$.

Here the least common multiple of the denominators is 12; hence the fractions may all be brought to twelfths.

15. Collect the fractions $\frac{x}{5}+\frac{x}{10}+\frac{x}{4}$. *Ans.* $\frac{11x}{20}$.

16. ... ... $\frac{x}{3}+\frac{x}{9}-\frac{x}{6}$. ... $\frac{5x}{18}$.

17. ... ... $x-\frac{x}{2}+\frac{x}{4}$. ... $\frac{3x}{4}$.

18. What is the difference between $\frac{x}{2}$ and $\frac{x}{3}$? *Ans.* $\frac{x}{6}$.

19. What is the difference between $\frac{2x}{5}$ and $\frac{x}{4}$? *Ans.* $\frac{3x}{20}$.

20. ... ... $\frac{7x}{9}$ and $\frac{2x}{3}$? ... $\frac{x}{9}$.

21. Reduce $\frac{5x}{3}$ to a mixed quantity.

$$\text{Here } \frac{5x}{3}=\frac{3x}{3}+\frac{2x}{3}=x+\frac{2x}{3}, \textit{ Ans.}$$

22. Reduce $\frac{9x}{2}$ to a mixed quantity. *Ans.* $4x+\frac{x}{2}$.

23. ... $\frac{8x}{3}$ ... ... $2x+\frac{2x}{3}$.

24. ... $\frac{7x}{5}$ ... ... $x+\frac{2x}{5}$.

25. Divide $\frac{x}{3}$ by $\frac{2}{3}$ *Ans.* $\frac{x}{2}$.

26. ... $\frac{2x}{15}$ by $\frac{4a}{5}$ ... $\frac{x}{6a}$.

24. PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. I paid 18*d.* for 55 pears and apples ; now 20 pears cost 6*d.*, and 5 apples cost 2*d.* How many of each did I purchase?

Let x=no. of apples,
then $55-x$=no. of pears.

Price of each apple $=\frac{2}{5}$ pence,

... ... pear $=\frac{6}{20}=\frac{3}{10}$ pence.

$\therefore$ Cost of the apples$=x$ times $\frac{2}{5}d.=\frac{2x}{5}$ pence,

And ... ... pears$=\frac{3}{10}(55-x)$ pence.

But the cost of the apples and pears together is 18*d.*

$$\therefore \frac{2x}{5}+\frac{3}{10}(55-x)=18.$$

Multiplying every term by 10, to get rid of the fractions,

$$4x+3\ (55-x)=180.$$

Performing the multiplication,

$$4x+165-3x=180,$$

$$\therefore\ x=15, \text{ no. apples.}$$

$$\text{And no. pears}=55-15=40.$$

2. A person bought oranges at 18*d.* per dozen; if he had bought 4 more for the same money, they would have cost 2*d.* a dozen less. How many did he purchase?

Let x=no. oranges,

then $x+4$= ... at 2*d.* less per dozen.

Price of each orange in the 1st case$=\frac{18}{12}=\frac{3}{2}d.$,

and ... ... 2nd ... $=\frac{16}{12}=\frac{4}{3}d.$

Now the data in the question will enable us to find two separate or independent values for the total cost of the oranges, thus,

Total cost $=\frac{3x}{2}$, but we have also,

Total cost $=\frac{4}{3}(x+4)$.

$$\therefore\ \frac{3x}{2}=\frac{4}{3}(x+4).$$

Solving this equation, we find $x=32$, the number required.

3. How many oranges can I get for 15*d.*, supposing one-third of them to be at 2*d.* each, and the remainder at $1\frac{1}{2}d.$?

Ans. 9.

4. Find a number such that if it be multiplied by 5, and 2 be taken from the product, one-half the remainder shall exceed the number by 5. *Ans.* 4.

5. A farmer lost one-third of his sheep; at another time he lost 4; he afterwards sold one-fourth of the remainder, and then the number of sheep left was 15. How many were there at first? *Ans.* 36.

6. A servant contracted for £9 a year and his livery. At the end of 5 months he was turned away, and then he received £2 and his livery. Required the value of the livery.

Let $x =$ the value of the livery in pounds,

then wages for 12 months $= 9 + x$,

$$\therefore \text{ wages for 1 month} = \frac{9+x}{12},$$

$$\therefore \textit{ wages for 5 months} = 5 \text{ times } \frac{9+x}{12} = \frac{45+5x}{12}.$$

But the *wages received for 5 months* $= 2 + x$,

$$\therefore \frac{45+5x}{12} = 2 + x.$$

Multiplying each side by 12, to get rid of the fraction,

$$45 + 5x = 24 + 12x,$$

$\therefore x = 3$ £, value of the livery.

7. A person paid a debt of £5 with sovereigns and half crowns; now there were half the number of sovereigns that there were half-crowns. How many were there of each?

Let $2x =$ the no. of half-crowns,

then $x =$ ... ... sovereigns.

Value of the half-crowns in shillings $= 2x \times 2\frac{1}{2} = 5x$,

... ... sovereigns ... ... $= 20x$.

But the sum of these two must amount to 100 shillings,

$$\therefore 5x + 20x = 100,$$

$\therefore x = 4$, no. sovereigns;

And the no. half crowns $= 2$ times $4 = 8$.

8. I received 13 pieces of money, composed of crowns and sixpences, in exchange for a sovereign. How many were there of each sort? *Ans.* 3 and 10.

9. A mass of copper and zinc weighs 30 lbs.; now the weight of the zinc in the mixture is two-thirds of the weight of copper. What weight is there of each?

Ans. 18 and 12 lbs.

10. One-third of a cask of beer is first drawn off, and afterwards one-half of the remainder, and then 14 gallons

are found remaining. How much did the cask at first contain? *Ans.* 42 gals.

11. A man and his wife lived in wedlock, one-third of his age, and one-fourth of hers. Now the man was 8 years older than his wife at marriage, and she survived him 20 years. Required their ages at marriage.

Let $x =$ the man's age at death, then,
the woman's age at death $= x - 8 + 20 = x + 12$.

But the question enables us to find two independent expressions for the time of wedlock, that is,

Time of wedlock = one-third man's age $= \frac{x}{3}$; but we have also,

Time of wedlock = one-fourth woman's age $= \frac{x+12}{4}$,

$$\therefore \frac{x}{3} = \frac{x+12}{4}.$$

Multiplying each side by 12, to get rid of the fractions,

$$4x = 3x + 36,$$

$$\therefore x = 36 \text{ years}.$$

Now the man lived one-third of this time in wedlock, that is 12 years; and therefore his age at marriage $= 36 - 12 = 24$ years; and then the woman's age $= 24 - 8 = 16$ years.

12. My son was born when I was 32 years of age; his age is now one-fifth of mine. What age is he? *Ans.* 8 years.

13. A gentleman gave 22 shillings among a certain number of persons; to three-fourths of them he gave 2*s.* each, and to the rest 5*s.* each. How many persons were there? *Ans.* 8.

14. A person rows 30 miles down a river and back in 7 hours; he rows 2 miles against the stream in the same time that he rows 5 miles with it. Find the rates of going and returning.

Let $2x =$ no. miles rowed per hour in returning.
$\therefore 5x =$ .. .., going.

Now the distance divided by the rate always gives us the time,

$$\therefore \frac{30}{2x} = \text{no. hours in returning,}$$

$$\text{and } \frac{30}{5x} = \quad \ldots \quad \ldots \quad \text{going.}$$

But the whole time in going and returning is 7 hours,

$$\therefore \frac{30}{2x} + \frac{30}{5x} = 7.$$

Multiplying each side by x, and bringing the fractions to whole numbers, we have,

$$15 + 6 = 7x,$$

$$\therefore x = 3.$$

And the rate in going $= 5 \times 3 = 15$ miles.

... ... returning $= 2 \times 3 = 6$ ...

15. A man travelled a certain journey at the rate of 4 miles an hour, and returned the distance at the rate of 3 miles an hour. He took 21 hours in going and returning. Required the total distance gone over. *Ans.* 72 miles.

16. Divide £30 into two parts, which shall have the ratio of 2 to 3. *Ans.* £12 and £18.

17. A person has £250 put out at interest in two separate sums, for one part he receives 3 per cent., and for the other 4 per cent. What must be the amount of each part, when the total annual interest is £8? *Ans.* £200 and £50.

18. A tradesman had a certain number of pounds of tea, for which he expected £4; but after selling 5 lbs., one-third the remainder was lost, so that he only realized £3 for the whole. How many pounds were there? *Ans.* 20.

19. When will the hands of a clock be together between 2 and 3 o'clock?

Here when the hour hand is at 2, the minute hand is at 12. Let $x =$ the no. of minutes, that the hour hand is past 2 o'clock. Now the minute hand goes 12 times as fast as the hour hand.

$\therefore 12x =$ the no. minutes, that the minute hand will have gone over.

But the minute hand will have gone over 10 minutes more than the hour hand.

$$\therefore\ 12x - x = 10;\ \text{and}\ \therefore x = \tfrac{10}{11},$$
$$\therefore\ \text{the time} = 10\tfrac{10}{11}\ \text{min. past 2.}$$

20. When will the hands be together between 3 and 4?

Ans. $16\frac{4}{11}$ min. past 3.

PRINCIPLES AND OPERATIONS.—SUBTRACTION.

25. Let us now see what we do when we have to subtract a quantity containing a minus term. For example, let it be required to subtract $7-3$ from 12. This is obviously the same as subtracting 4 from 12. But let us perform the operation so that the different quantities may appear in the result.

From 12
Subtract $7-3$
Remainder $12-7+3=8$.

Taking 7 from 12, the result will be expressed by $12-7$; but this will be too little for the remainder, for we have taken away 3 too much; therefore the true remainder will be found by adding 3 to $12-7$, that is, the true remainder will be $12-7+3$. Here we observe, that the terms to be subtracted are written in the remainder with their signs changed.

Or generally, let it be required to subtract $c-e$ from a.

From a
Subtract $c-e$
Remainder $a-c+e$.

Here we have not to subtract c units from a units, but c units less by e units. By taking c from a we have the remainder $a-c$; but this result is too little, for we have taken away e units too much; therefore the true remainder will be obtained by adding e to $a-c$, that is, the remainder will be $a-c+e$.

The following questions may be put by the teacher in the course of the demonstration:—

Teacher. (Writing down $a - c$). What have I taken away from a?

Pupil. You have subtracted c.

Teacher. Have I taken away too much or too little?

Pupil. You have taken away too much.

Teacher. How much too much have I taken away?

Pupil. You have taken away e too much.

Teacher. What must I then do to make this the correct answer?

Pupil. You must add an e.

Teacher. What have you to say about the signs of the quantities to be subtracted?

In like manner we have, $2x-(x-4)=x+4$; because by taking x away from $2x$ our result would have been too little by 4.

Hence the rule of subtraction in algebra is this,—*Change the signs of the quantities to be subtracted, and then collect the quantities as in addition.*

As an illustration of this principle,—let us suppose a person to have a certain amount of *property*, and that he owes a certain amount of *debt*, of which there has been a certain sum of *money paid*, then the sum he is actually owing will be, *debt—money paid*, and his actual wealth will be,

property — (debt — money paid) = property — debt + money paid.

Obs. The first term of any expression is always to be taken plus, when no sign is prefixed. Thus $x-a$ is intended to mean $+x-a$.

EXAMPLES IN SUBTRACTION.

1.	From $5x$	subtract $2x-7$.	*Ans.*	$3x+7$.
2.	... $6x$	... $3x-a$.	...	$3x+a$.
3.	... $2x+a$	... $x-2a$.	...	$x+3a$.
4.	... $x-2a$	... $x-3a$.	...	a.

5. Find the value of $2x-(x-3a)$. *Ans.* $x+3a$.

6. ... ... $4x+2-(3x-4)$. ... $x+6$.

7. ... ... $3a-1-(a-6)$. ... $2a+5$.

8. What is the difference between $\frac{x}{2}$ and $\frac{x-3}{3}$?

Here bringing the fractions to the same denominator, and subtracting the one numerator from the other,

$$\frac{3x}{6}-\frac{2x-6}{6}=\frac{x+6}{6}. \quad \textit{Ans.}$$

9. Subtract $\frac{2x-4}{4}$ from $\frac{4x-3}{3}$.

Bringing both fractions to twelfths, we have,

$$\frac{16x-12}{12}-\frac{6x-12}{12};$$

taking the latter numerator from the former, by changing the signs, the difference becomes,

$$\frac{16x-12-(6x-12)}{12}=\frac{10x}{12}=\frac{5x}{6}. \quad \textit{Ans.}$$

10. Subtract $\frac{a-x+c}{2}$ from a.

Bringing the quantities to the same denominator, and changing the signs of the numerator to be subtracted,

$$\frac{2a}{2}-\frac{a-x+c}{2}=\frac{2a-(a-x+c)}{2}=\frac{a+x-c}{2}. \quad \textit{Ans.}$$

Here it is important to observe, that the minus sign prefixed to the fraction indicates that the *whole* of the numerator is to be subtracted.

11. Subtract $\frac{x-2}{3}$ from $\frac{x-3}{2}$. *Ans.* $\frac{x-5}{6}$.

12. ... $\frac{2x-1}{2}$ from $3x$. ... $\frac{4x+1}{2}$.

13. ... $\frac{x+a-2}{3}$ from $\frac{9x}{5}$. ... $\frac{22x-5a+10}{15}$.

14. Find the value of $x-\frac{2x-1}{5}$. ... $\frac{3x+1}{5}$.

15. Find the value of $\frac{x+1}{2}-\frac{2x-3}{4}$. *Ans.* $\frac{5}{4}$.

16. ... $3a-\frac{a-c+2}{3}$. ... $\frac{8a+c-2}{3}$.

26. PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. If 2 and 6 be separately taken from a certain number, the first difference will be as much greater than 8, as the second difference is less than 8. Required the number.

Let $x =$ the number, then

$x-2 =$ two taken from the number

$x-6 =$ six ... ...

Hence we have by the question,

$$(x-2)-8=8-(x-6),$$

Performing the subtraction and collecting,

$$x-10=14-x,$$

$$\therefore x=12, \text{ the number.}$$

2. If 2 be taken from a certain number; then one-half the remainder taken from the number itself will be equal to 8. Required the number.

Here, we find the equation,

$$x-\frac{x-2}{2}=8.$$

Multiplying every term by 2, to get rid of the fraction,

$$2x-(x-2)=16;$$

Performing the subtraction, by changing the signs,

$$2x-x+2=16,$$

$$\therefore x=14, \text{ the number.}$$

3. There is a number, such that if 3 be taken from it, then one-fifth the remainder subtracted from 7 will be equal to 4. Required the number. *Ans.* 18.

4. A party took a boat from Chelsea to Richmond at 6*d.* each, with this condition, that for every person taken in by the way, 4*d.* should be deducted from the joint fare. Now the number, taken in by the way, was 3 less than the

number of passengers at first, who then only pay 5*d.* each. How many were there at first?

Let $x =$ the no. at first,

then $x-3 =$ the no. taken in by the way;

$\therefore\ 6x =$ cost of the first party if none had been taken in by the way,

and $(x-3)\,4 =$ deduction arising from the no. taken in;

but $5x =$ fare actually paid.

Hence we have the equation,

$6x-(x-3)\,4 = 5x$; solving this equation we find,

$x = 4$, the number.

5. A person first spent £10. of his money, and afterwards one-fourth of the remainder, and then he found that he had £30 left. How much had he at first? *Ans.* £50.

6. A farmer bought a certain number of sheep; if he had paid 2*s.* a head more, they would have cost 60*s.*; but if he had paid 3*s.* a head less, they would have cost 35*s.* How much did he pay for each?

Let $x =$ the no. shillings each sheep cost.

Now the data of the question will enable us to find two independent expressions for the number of sheep.

$$\text{Thus, } \frac{\text{cost whole no. sheep}}{\text{cost each sheep}} = \text{no. sheep};$$

$$\therefore\ \frac{60}{x+2} = \textit{no. sheep},$$

$$\text{and } \frac{35}{x-3} = \textit{no. sheep}.$$

$$\therefore\ \frac{60}{x+2} = \frac{35}{x-3}.$$

Clearing of fractions, by first multiplying by $x+2$, and then by $x-3$, we have,

$$60x-180 = 35x+70,$$

$$\therefore\ x = 10, \text{ the no. sheep.}$$

7. If a labourer were paid 1*s.* a day less than he really has, his wages would be 10*s.*; but if he were paid 1*s.* a day more, his wages would be 20*s.* Required his wages per day. *Ans.* 3*s.*

8. A man by working for 5 days, received 10*s.* more than he would have done, if he had wrought for 3 days at 1*s.* per day less. Required his wages per day. *Ans.* 3*s.* 6*d.*

27. EXAMPLES IN NUMERICAL EQUATIONS.

1. Find the value of x in the equation,

$$\frac{3x-2}{2}-\frac{2x-3}{3}=5.$$

Multiplying every term by 6, to clear the fractions,

$$3\,(3x-2)-2\,(2x-3)=30.$$

Performing the multiplication and subtraction, observing to change the signs of the quantities before which the minus sign is placed,

$$9x-6-4x+6=30,$$

Collecting, $5x=30,$

and $x=6$

2. $\frac{x-4}{3}-\frac{x-1}{4}=1.$ *Ans.* $x=25.$

3. $x-\frac{3x-5}{4}=2.$ ... $x=3.$

4. $5\,(x+2)-\frac{x-3}{2}=52.$ ... $x=9.$

5. $2x-5\,(2-x)+2=6.$ ... $x=2.$

6. $\frac{3x+8}{5}-\frac{x-8}{4}-2=3.$ ... $x=4.$

7. $\frac{5x+3}{2}-2\,(x-2)=6.$ ... $x=1.$

PRINCIPLES AND OPERATIONS.—MULTIPLICATION, ETC.

28. *Multiplication when there are two or more terms in the multiplier.* To multiply $a+b$ by $c+d$, we have $a+b$ to take c times, and then d times, and the sum of these will be $a+b$ taken $(c+d)$ times. Thus,

$$(a+b)\times(c+d)=c\text{ times }(a+b)+d\text{ times }(a+b)$$
$$=ac+bc+ad+bd.$$

Here we first multiply $a+b$ by c, and then by d, for the whole product.

To illustrate the principle upon which this process depends: let us take any quantity, say a *bag* of nuts, and suppose that I give c *bags* to one boy, and d *bags* to another; then it is easy to see, that I should have given $(c+d)$ *bags* away. Now this is only in other words stating that,

$$c \text{ bags} + d \text{ bags} = (c+d) \text{ bags};$$

and if there were $a+b$ nuts in each bag, we should have,

$$c \text{ times } (a+b) + d \text{ times } (a+b) = (c+d) \text{ times } (a+b).$$

Let it now be required to multiply $7-4$ by $8-2$. This will obviously be the same as multiplying 3 by 6; but we purpose to go through the process, so as to retain the different terms in the product.

$$\begin{aligned}(7-4)\times(8-2) &= 8 \text{ times } (7-4) - 2 \text{ times } (7-4)\\ &= 7\times 8 - 4\times 8 - 7\times 2 + 4\times 2 = 18.\end{aligned}$$

Here $7-4$ is to be taken 2 times less than 8 times.

Multiplying $7-4$ by 8, takes it 2 times too many. We therefore multiply $7-4$ by 8, and then subtract $7-4$ taken 2 times, in order to obtain the true product.

Or generally, let it be required to multiply $(a-b)$ by $(c-d)$.

$$\begin{array}{lrl} \text{Multiply} & a-b & \\ \text{by} & \underline{c-d} & \\ & ac-bc & = c \text{ times } (a-b) \\ & \underline{-ad+bd} & = d \text{ times } (a-b) \text{ taken away.} \\ & ac-bc-ad+bd & \text{the product.} \end{array}$$

Here $a-b$ is not to be taken c times, but d times less than c times. First then taking c times $(a-b)$, we know that this result is too much, for $a-b$ has been taken d times too many; therefore the correct product will be found by subtracting d times $(a-b)$ from the preceding result.

The following questions may be put by the teacher in the course of the demonstration.

Teacher. (After having written down the product $ac - bc$). How many times have I here taken the multiplicand?

Pupil. You have taken it c times.

Teacher. Is this the correct product?

Pupil. No. It is too much.

Teacher. How many times too much have I taken $a-b$?

Pupil. You have taken it d times too much.

Teacher. What must I do to get the true product?

Pupil. You must take d times $a-b$ and subtract the result from what you have written.

In these examples it will readily be seen, that + multiplied by + gives + for the product; − multiplied by − gives +; + by − gives −; and − by + gives −; or as the rule is generally expressed, *like* signs multiplied together produce *plus*, and *unlike* signs multiplied together produce *minus*.

29. EXAMPLES IN MULTIPLICATION.

1. Multiply $2x+3$ by $2x-3$

$$\begin{array}{r} 2x+3 \\ \underline{2x-3} \\ 4x^2+6x \quad\quad \\ \underline{\quad\quad -6x-9} \\ 4x^2 \quad\quad * -9 \end{array}$$

Here we first multiply by $2x$, and then by 3, observing that *like signs* give a *positive*, and *unlike signs* a *negative* result. Upon adding the product it will be seen that $-6x$ destroy $+6x$.

2. Multiply	$2x-4$ by $3x-2$.	*Ans.*	$6x^2-16x+8$.
3. ...	$5a+3$ by $3a-5$.	...	$15a^2-16a-15$.
4. ...	$2x+3a$ by $4x-3a$.	...	$8x^2+6ax-9a^2$.
5. ...	$4x+2a$ by $2x-a$.	...	$8x^2-2a^2$.
6. ...	x^2+xy+y^2 by $x-y$.	...	x^3-y^3.
7. ...	$\frac{2x+1}{3}$ by $\frac{1-2x}{2}$.	...	$\frac{1-4x^2}{6}$.

Here by Art. 20. we multiply the numerators together and the denominators together.

8. Multiply $\frac{3a+2}{5}$ by $\frac{5}{a}$. *Ans.* $\frac{3a+2}{a}$.

9. ... $\frac{a-b}{2}$ by $\frac{a+b}{2}$. ... $\frac{a^2-b^2}{4}$.

30. *Powers and Roots.* Let it be required to multiply a^4 by a^3. Now, a^4 means that there are four a's multiplied together, that is, $a \times a \times a \times a$; and a^3 that there are three a's multiplied together, that is, $a \times a \times a$; hence we have, $a^4 \times a^3 = \overline{a \times a \times a \times a} \times \overline{a \times a \times a} = a^7$. Here it will be observed, that we *add the exponents of the factors to obtain the exponent of the product.* Let it be required to divide a^5 by a^2. Here we have, $\frac{a^5}{a^2} = \frac{a \times a \times a \times a \times a}{a \times a} = a \times a \times a = a^3$. In this case it will be seen that we *subtract the exponent of the divisor from the exponent of the dividend to obtain the exponent of the quotient.*

Thus, we have generally,

$$\frac{a^n}{a^m} = a^{n-m} \ldots\ldots (1)$$

Let $m=n$ in this expression, then,

$$\frac{a^n}{a^n} = a^{n-n}, \therefore a^0 = 1;$$

that is, any quantity having 0 as an exponent, gives unity. Again, let $n=0$ in expression (1), then

$$\frac{a^0}{a^m} = a^{-m}, \therefore \frac{1}{a^m} = a^{-m},$$

that is, any quantity may be taken from the denominator to the numerator by simply changing the sign of the exponent, and conversely. Thus, $\frac{b}{a^2} = ba^{-2}$, and $ca^{-3} = \frac{c}{a^3}$.

Let it now be required to find the fourth power of a^2. Here the fourth power of a^2, or $(a^2)^4 = a^2 \times a^2 \times a^2 \times a^2 = a^8$, by adding the exponents. In this case, the number representing the proposed power is multiplied into the exponent of the a.

31. By the square root of any quantity, we mean the finding of a number, which, multiplied by itself, will produce the proposed quantity: thus, $\sqrt{9}\times\sqrt{9}=9$, hence it follows that $\sqrt{9}=3$, because $3\times3=9$. In like manner $\sqrt{x}\times\sqrt{x}=x$. Hence it follows, that the square of $\sqrt{x}=\sqrt{x}\times\sqrt{x}=x$; and the square of $\sqrt{a+x}=\sqrt{a+x}\times\sqrt{a+x}=a+x$.

By the third root, or cube root, of any quantity, we mean the finding of a number, which, multiplied by itself three times, will produce the proposed quantity: thus, $\sqrt[3]{8}\times\sqrt[3]{8}\times\sqrt[3]{8}=8$, hence it follows that $\sqrt[3]{8}=2$, because $2\times2\times2=8$. In like manner $\sqrt[3]{x}\times\sqrt[3]{x}\times\sqrt[3]{x}=x$. Hence it follows, that the third power, or cube, of $\sqrt[3]{x}=\sqrt[3]{x}\times\sqrt[3]{x}\times\sqrt[3]{x}=x$; and the third power of $\sqrt[3]{c+x}=\sqrt[3]{c+x}\times\sqrt[3]{c+x}\times\sqrt[3]{c+x}=c+x$. And so on to other cases of roots.

The symbol $\sqrt{x}$ may also be written $x^{\frac{1}{2}}$, because $x^{\frac{1}{2}}\times x^{\frac{1}{2}}=x$, by the addition of the exponents, Art. 30. In like manner $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$, because $x^{\frac{1}{3}}\times x^{\frac{1}{3}}\times x^{\frac{1}{3}}=x$, by the addition of the exponents. And so on to other cases of roots.

32. *Division when there are two or more terms in the divisor.* Division being the reverse operation of multiplication, the divisor multiplied by the quotient, must give the dividend. Thus, to divide $6a^2$ by $2a$, is the same as finding what quantity must be multiplied by $2a$ to give $6a^2$ for the product; in fact, $6a^2\div2a=3a$, because $2a\times3a=6a^2$. In like manner, if we have $8x^2-16x+6$ to divide by $2x-3$, the quotient required multiplied by this divisor, must give us the proposed dividend. The operation, therefore, may be performed after the following manner.

$$
\begin{array}{l}
2x-3)\ 8x^2-16x+6\ (4x-2 \\
\qquad\quad \underline{8x^2-12x} \qquad =4x \text{ times } (2x-3) \\
\qquad\qquad\quad -4x+6 \\
\qquad\qquad\quad \underline{-4x+6}=-2 \text{ times } (2x-3.) \\
\qquad\qquad\quad \underline{\ \cdots\ }
\end{array}
$$

Here we find how many times the first term of the divisor is

contained in the first term of the dividend, that is, $8x^2$ divided by $2x$ gives $4x$ for the quotient; then multiplying the divisor by $4x$, we obtain $8x^2-12x$, which, subtracted from the dividend, leaves the remainder $-4x+6$. Again, proceeding in the same way, $-4x$ divided by $2x$ gives -2 for the quotient, then multiplying the divisor by -2, we obtain $-4x+6$, which, subtracted from the remainder, leaves nothing.

To test the accuracy of the result, multiply the divisor and quotient together; and the product will be the dividend. Before commencing the division, in any question, the terms of the dividend and divisor must be arranged according to the *powers* of some letter common to both.

33. EXAMPLES IN DIVISION.

1.	Divide x^2-6x+8 by $x-2$.	*Ans.* $x-4$.
2.	... $6x^2+5x+1$ by $2x+1$.	... $3x+1$.
3.	... $5x^2+31x+30$ by $5x+6$.	... $x+5$.
4.	... $2x^2+5ax+3a^2$ by $2x+3a$.	... $x+a$.
5.	... $9x^2-4a^2$ by $3x-2a$.	... $3x+2a$.
6.	... $x^2+(a-2)\,x-2a$ by $x-2$.	... $x+a$.
7.	... $4x^2-12x+9$ by $2x-3$.	... $2x-3$.

USEFUL THEOREMS.

34. The following theorems ought to be committed to memory.

$$(a+b)\times(a-b)=a^2-b^2.$$

That is, *the product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus we have,

$$(2x+3a)\times(2x-3a)=4x^2-9a^2.$$
$$(x+1)\times(x-1)=x^2-1.$$

And so on to other cases.

35. When a quantity is multiplied by itself it gives the second power of that quantity, thus,

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline (a+b)^2=a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline (a-b)^2=a^2-2ab+b^2 \end{array}$$

Multiplying the first by $a+b$, and the second by $a-b$, we obtain the third powers of these quantities, that is,

$$(a+b)^3=a^3+3a^2\,b+3a\,b^2+b^3$$
$$(a-b)^3=a^3-3a^2\,b+3a\,b^2-b^3.$$

Again multiplying the first of the two preceding results by $a+b$, and the second by $a-b$, we obtain the fourth powers of these quantities; that is,

$$(a+b)^4=a^4+4a^3\,b+6a^2\,b^2+4a\,b^3+b^4$$
$$(a-b)^4=a^4-4a^3\,b+6a^2\,b^2-4a\,b^3+b^4.$$

And so on to other powers. The student is recommended to work these powers fully out.

The right hand expression is called the development or expansion, of a *binomial*, that is, an expression consisting of two terms. The coefficients and powers in the development, may be derived from the given power by a general law, which we now proceed to explain. Taking the fourth power, or $(a+b)^4$, as an example.

The first term in the development is a raised to the 4th power, and in every succeeding term the power of a is decreased by one. In the second term b appears in the 1st power, and in every succeeding term the power of b is increased by one, until we arrive at the 4th power. The coefficient of the second term is 4, the same number as the given power. The coefficient of the third term is 6, and it is obtained by multiplying the coefficient of the preceding term by the exponent of the a, and dividing this product by 2, the number of this term; thus $\frac{4\times 3}{2}=6$. The coefficient of the fourth term is 4, and it is obtained after the same manner, that is, by multiplying the coefficient of the preceding term by the exponent of the a, and dividing the product by 3, the number of this term; thus $\frac{6\times 2}{3}=4$. And so on to the remaining term. When b is minus, the development is the same as when it is plus, excepting that the terms are alternately plus and minus. This is called the binomial theorem.

It is very useful in algebra, as it enables us to write down any power of a binomial without actually going over the calculation. Thus for example,

$$(x+2b)^3 = x^3+3x^2\times 2b+3x\times(2b)^2+(2b)^3$$
$$= x^3+6b\,x^2+12b^2\,x+8b^3.$$

1°. The general form of the expansion of a binomial is expressed by,

$$\left(a+b\right)^n = a^n+n\,a^{n-1}\,b+\frac{n\,(n-1)}{1.\,2}\,a^{n-2}b^2+$$
$$\frac{n(n-1)(n-2)}{1.\,2.\,3}a^{n-3}b^3+\&c.$$

2°. The signs of the even powers of a minus quantity will be plus; and the signs of the odd powers will be minus. Thus, from the rule for the multiplication of signs, we have,

the 2nd power of $-a = -a\times -a = +a^2$
... 3rd ... of $-a=-a\times -a\times -a=-a^3$
... 4th ... of $-a=-a\times -a\times -a\times -a=a^4$

And so on to other powers.

3°. The even roots of a plus quantity may be either plus or minus; and the odd roots of a minus quantity will be minus.

Thus, the square root of a^2, may be either $+a$ or $-a$, or as this double sign is expressed, $\pm a$; because $-a\times -a=a^2$, and also $+a\times +a=a^2$.

Thus, the cube root of $-a^3=-a$; because $-a\times -a\times -a=-a^3$.

EXAMPLES IN POWERS AND ROOTS.

1. What is the 3rd power of b^2?			*Ans.*	b^6.
2. ...	4th ...	of a^3?	...	a^{12}.
3. ...	2nd ...	of $2+x$?	...	$4+4x+x^2$.
4. ...	2nd ...	of $x+2a$?	...	$x^2+4ax+4a^2$.
5. ...	3rd ...	of $-2x$?	...	$-8x^3$.
6. ...	2nd ...	of $-3x^2$?	...	$9x^4$.

7. What is the 2nd power of $\frac{2x}{3}$? *Ans.* $\frac{4x^2}{9}$.

8. ... 3rd ... of $v+\frac{1}{v}$?

Here by the law of the expansion of a binomial,

$$\left(v+\frac{1}{v}\right)^3=v^3+3v^2\,\frac{1}{v}+3v\left(\frac{1}{v}\right)^2+\left(\frac{1}{v}\right)^3$$

$$=v^3+3v+\frac{3}{v}+\frac{1}{v^3}.$$

The same result will, of course, be obtained by multiplying $v+\frac{1}{v}$ by itself three times.

9. What is the 3rd power of $v+\frac{2}{v}$?

Ans. $v^3+6v+\frac{12}{v}+\frac{8}{v^3}$.

10. What is the 2nd power of $x+\frac{1}{x}$? *Ans.* $x^2+2+\frac{1}{x^2}$.

11. ... 4th ... of $\sqrt{x}$? ... x^2.

12. ... 2nd ... of $\sqrt{x}+2$? ... $x+4\sqrt{x}+4$.

13. What is the 3rd or cube root of $8x^6$? *Ans.* $2x^2$.

Because, $2x^2\times 2x^2\times 2x^2=8x^6$.

14. What is the 2nd root of x^8? *Ans.* x^4.

15. ... 3rd ... of x^{12}? ... x^4.

16. ... 2nd ... of $9x^2$? ... $3x$.

17. ... 2nd ... of $16x^4$? ... $4x^2$.

18. What is the square root of $x^2+2xy+y^2$? ... $x+y$. Because $(x+y)$ multiplied by itself gives the proposed expression.

19. What is the square root of x^2+6x+9? *Ans.* $x+3$. Now we know, from the *form* of the square of a binomial, that if this quantity has an exact root, then this root must be $x+3$; where x is the square root of the x^2, 3 is the square root of the 9. It now remains for us to see whether or not the middle term, $6x$, is that of a square quantity; for this purpose we have the test, — twice the product of x and

$3 = x \times 3 \times 2 = 6x$, which, being the same as the middle term of the given expression, we conclude that it is a square quantity and that its root is $x+3$.

20. What is the square root of $x^2+8x+16$? *Ans.* $x+4$.
21. ... ... of x^2-2x+1? ... $x-1$.
22. ... ... of x^2-4x+4? ... $x-2$.
23. ... ... of $4x^2-4x+1$? ... $2x-1$.
24. What is the cube root of $(x+y)^3$? ... $x+y$.

36. EXAMPLES IN SIMPLE EQUATIONS.

1. Find the value of x in the equation,

$$\frac{4x+2}{x-1}+\frac{2x-4}{2x-2}=6.$$

Here it will be observed that the denominator $2x-2=2(x-1)$; by multiplying, therefore, every term by this quantity, we shall clear the equation of fractions;

$$\therefore\ 8x+4+2x-4 = 12x-12,$$
$$\therefore\ x=6.$$

2. Find the value of x in the equation,

$$\frac{2x+3}{x-1}+\frac{x-17}{x^2-1}=2.$$

Here, to clear the equation of fractions, we multiply by x^2-1, observing that as this quantity (see Art. 34.) is equal to $(x+1)\times(x-1)$, the numerator of the first fraction will be multiplied by $x+1$.

$$\therefore\ 2x^2+5x+3+x-17=2x^2-2.$$

Here the $2x^2$ on each side of the equation will destroy each other, and then we readily find, $x=2$.

3. $x+2=\dfrac{x^2+16}{x+2}$ *Ans.* $x=3$.

4. $3x-2=\dfrac{6x^2+1}{2x-3}+8.$ ... $x=1$.

5. $\dfrac{3x-2}{x+1}+\dfrac{9x-21}{x^2-1}=3.$ ... $x=4$.

6. $\frac{3}{x+1}+\frac{4}{2x+2}+\frac{3x-3}{x^2-1}=1.$ *Ans.* $x=7.$

7. $\frac{2x^2+13x}{x+4}=x+9.$

Here, multiplying each side by $x+4$, we have,

$$2x^2+13x=x^2+13x+36,$$

$$\therefore x^2=36.$$

And taking the square root of each side of the equation,

$$x=6.$$

8. $\frac{5x^2+25x}{x+9}=x+16.$ *Ans.* $x=6.$

9. $\sqrt{x^2+9}=5.$

Here, squaring each side (see Art. 31.), we find,

$$x^2+9=25,$$

$$\therefore x^2=16.$$

Taking the square root of each side,

$$x=4.$$

10. $\sqrt{3x+4}=5.$ *Ans.* $x=7.$

11. $\sqrt{4x^2+16}=5.$ ... $x=\frac{3}{2}.$

12. $\sqrt{x^2+6x-4}=x+2.$

Here, squaring each side, in order to clear the equation of the square root;

$$x^2+6x-4=x^2+4x+4,$$

$$\therefore 2x=8, \text{ and } x=4.$$

13. $\sqrt{x^2+8x-5}=x+3.$ *Ans.* $x=7.$

14. $\sqrt{2x^2+4x-5}=x+2.$ ... $x=3.$

15. $\frac{\sqrt{x}}{2}+8=\frac{\sqrt{x}}{4}+9.$ ... $x=16.$

First subtracting 8 from each side, and then multiplying by 4 to clear the equation of fractions, we have,

$$2\sqrt{x}=\sqrt{x}+4.$$

Subtracting $\sqrt{x}$ from each side,

$$\sqrt{x}=4.$$

Squaring each side, $x=16.$

16. $\sqrt{x}-4=\frac{\sqrt{x}-}{3}-\frac{\sqrt{x}-5}{12}$. *Ans.* $x=25$.

17. Given, $a\,x+b=2b+d$, to find x.
Taking b from each side of the equality,

$$ax=b+d;$$

Dividing by a, we have,

$$x=\frac{b+d}{a}.$$

18. Given, $ax+bx=cx+d$, to find x.
Taking cx from each side,

$$ax+bx-cx=d;$$

Collecting the x's, as explained in Art. **12.**,

$$(a+b-c)x=d,$$

Dividing each side by $a+b-c$, we have,

$$x=\frac{d}{a+b-c}.$$

19. $x=\frac{bx+a}{c}$ *Ans.* $x=\frac{a}{c-b}$.

20. $\frac{a}{x}+\frac{b}{x}=d.$ ... $x=\frac{a+b}{d}$.

21. $(x+a)^2-a^2=b+2ax.$ ... $x=\sqrt{b}$.

22. $\frac{x}{a}+\frac{x}{b}=1.$ ... $x=\frac{ab}{a+b}$.

37. EXAMPLES IN EQUATIONS WITH TWO UNKNOWN QUANTITIES.

There are three general methods, or artifices, used in solving equations of two unknown quantities; but the object of all of them consists in reducing the *two* equations into *one* containing only one of the unknown quantities, then this equation may be solved by the methods already given.

1. Given the two following equations to find the values of x and y.

$$2x+5y=26 \;\ldots\; (1.)$$
$$5x+6y=39 \;\ldots\; (2.)$$

In order to make the number of x's the same in each equation, we multiply the first by 5, and the second by 2; then we have,

$$10x+25y = 130$$
$$10x+12y = 78.$$

Subtracting the second of these equations from the first, in order to destroy the x's, we have,

$$13y = 52,$$
$$\therefore\ y = 4.$$

Having found the value of y to be 4, we now put, or *substitute*, this 4 in the place of y, in the first equation, and then we have,

$$2x+5\times 4 = 26,$$
$$\therefore\ x = 3.$$

By putting these values for x and y, in the two given equations, we are enabled to verify the results, or to show that the values have been correctly found.

2. $\left.\begin{array}{r} 3x+2y = 7 \\ 4x+5y = 14 \end{array}\right\}$; *Ans.* $x = 1$. ... $y = 2$.

3. $\left.\begin{array}{r} 2x-3y = 4 \\ 3x+2y = 32 \end{array}\right\}$; ... $x = 8$. ... $y = 4$.

4. $\left.\begin{array}{r} 7x-4y = 31 \\ x+4y = 9 \end{array}\right\}$; ... $x = 5$. ... $y = 1$.

Here the y's will be destroyed or eliminated by simply adding these equations.

38. By the method given in Ex. 1., any equation of this kind may be solved; but we shall give another method in the solution of the following question.

1. $$2x-5y = 2 \ \ldots\ (1.)$$
$$5x+3y = 36 \ \ldots\ (2.)$$

Find the value of x from the first.

$$x = \frac{2+5y}{2}.$$

Find the value of x from the second.

$$x = \frac{36-3y}{5}.$$

These two values for the same thing, must be equal to each other;

$$\therefore \frac{2+5y}{2} = \frac{36-3y}{5}$$

Having now got an equation containing only *one* unknown quantity, we readily find $y = 2$; then putting this number for y in either of the expressions for x, we find $x = 6$.

2. $\left.\begin{aligned} x-y &= 1 \\ 7x-6y &= 15 \end{aligned}\right\};$ *Ans.* $x = 9$. ... $y = 8$.

3. $\left.\begin{aligned} 3x+y &= 20 \\ x-2y &= 2 \end{aligned}\right\};$ ... $x = 6$. ... $y = 2$.

4. $\left.\begin{aligned} \frac{x}{2}-3y &= 3 \\ \frac{x}{3}+2y &= 6 \end{aligned}\right\};$ ... $x = 12$. ... $y = 1$.

Here we multiply the first equation by 2, and the second by 3, in order to get rid of the fractions; and then we proceed with the two resulting equations after the methods already described.

5. $\left.\begin{aligned} \frac{x}{5}+3y &= 11 \\ x+2y &= 16 \end{aligned}\right\};$ *Ans.* $x = 10$. ... $y = 3$.

39. Besides the two methods which have been given, there is a third, called the method of *substitution*, which consists in *substituting* the value of x found from one of the equations, for x in the other equation.

1. $2x-4y = 1+x$ ... (1.)
   $3x+5y = 20$ ... (2.)

Find the value of x from the first.

$$x = 1+4y,$$

Substituting this value of x in equation (2.),

$$3(1+4y)+5y = 20.$$

From this equation we find $y = 1$; then substituting this value of y, for y in the expression for x, we find,

$$x = 1+4\times 1 = 5.$$

2. $\left.\begin{array}{l} \frac{x}{2}+4y = 10+y \\ 3x+7y = 38 \end{array}\right\};$ *Ans.* $x = 8.$ ... $y = 2.$

3. $\left.\begin{array}{l} \frac{x}{5}-2y+4 = 1 \\ \frac{x}{3}+\frac{6y}{9} = 7 \end{array}\right\};$ ... $x = 15.$ ... $y = 3.$

4. $\left.\begin{array}{l} \frac{x+1}{2}+3y = 24 \\ 5x+\frac{2y+1}{3} = 30 \end{array}\right\};$ ... $x = 5.$ ... $y = 7.$

5. $\left.\begin{array}{l} 3x+8 = x+2y+22 \\ 7x-5y = 53 \end{array}\right\};$ ... $x = 9.$ ... $y = 2.$

40. PROBLEMS IN EQUATIONS OF TWO UNKNOWN QUANTITIES.

1. 7 Ducks and 3 pigeons cost 17*s.*; and 5 ducks and 2 pigeons cost 12*s.*; how much did each cost?

Let x = cost of each duck in shillings,
and y = ... pigeon ..

$\therefore$ Cost of the first purchase in shillings $= 7x+3y$,
and ... second ... $= 5x+2y$.

But the first is equal to 17*s.*, and the second to 12*s.*; hence we have the two equations,

$$7x+3y = 17 \ \ldots \ (1.)$$
$$5x+2y = 12 \ \ldots \ (2.)$$

Solving these equations by any of the methods already described, we find, x = 2*s.*, and y = 1*s.*

2. A farmer sold 5 sheep and 3 lambs, for £3 12*s.*; and afterwards, at the same rate each, 3 sheep and 5 lambs, for £2 16*s.* Required the cost of each. *Ans.* 12*s.* and 4*s.*

3. I bought a certain number of lbs. of tea at 5*s.* per lb., and a certain number of lbs. of coffee at 2*s.* per lb., and paid for the whole 22*s.*; now if I had paid 1*s.* a lb. more for

each, I should have paid 30*s*. for the whole. How many lbs. were there of each? *Ans.* 2 lbs. and 6 lbs.

4. John says to James, if you give me 2*d*., I shall then have the same money as you; but if I give you 2*d*., you will have three times as much as I then will have. How much had each? *Ans.* 10*d*. and 6*d*.

Let $x =$ no. pence John had,
and $y =$ ... James had;
then, $x+2 =$ John's money after having got 2*d*.
and $y-2 =$ James' ... given 2*d*.

But these quantities, by the question, are equal,

$$\therefore\ x+2=y-2 \ \ldots\ (1.)$$

Again, to obtain another equation, we have,

$x-2 =$ John's money after having given 2*d*.,
and $y+2 =$ James' ... got 2*d*.

But by the question, the latter quantity is 3 times the former,

$$\therefore\ y+2=3(x-2) \ \ldots\ (2.)$$

Then from the equations (1.) and (2.) the values of x and y are readily found.

5. If James had 6 more marbles than he has, he would have twice as many as Thomas; and if Thomas had 9 more, he would have twice as many as James. How many has each? *Ans.* 8 and 7.

6. Five years hence, my son's age will be one-fourth of mine; and 15 years hence his age will be two-fifths of mine. What are our ages? *Ans.* 5 and 35 years.

7. A and B have together £70; A loses the third part of his money, and B the fourth part, and then B has £10 more than A. How much has each?

In order to avoid fractions,

Let $3x =$ A's money,
and $4y =$ B's ...

then, A's money after losing one-third $= 3x - x = 2x$.
B's ... ... one-fourth $= 4y - y = 3y$.

But the latter is £10 more than the former,

$$\therefore\ 3y = 2x + 10 \ \dots\ (1.)$$

Now A and B's money make up £70, hence we obtain another equation,

$$3x + 4y = 70 \ \dots\ (2.)$$

Subtracting 4 times the first, from 3 times the second, we obtain,

$$9x = 170 - 8x,$$

$$\therefore\ x = 10$$

$\therefore$ A's money $= 3x =$ £30; and B's $= 70 - 30 =$ £40.

8. A grocer wishes to mix 25 lbs. of sugar at 6*d.* a lb., with two sorts of sugar, one at 4*d.* a lb., and the other at 8*d.*; how many lbs. of each sort must he use, so that 70 lbs. of the mixture may be worth 7*d.* a lb.?

Ans. 5 lbs. and 40 lbs.

9. I am 38 years of age; now if my age be put to Peter's, the sum will be 4 times John's age; but if my age be put to John's, the sum will be 5 times Peter's age. What are Peter and John's ages? *Ans.* 10 and 12 years.

10. There is a number consisting of two digits (or places of figures) which is equal to twice the sum of the digits; and if 9 be added to the number, the sum will be one-third the number with its digits inverted. What is the number?

Let $x =$ the digit in the ten's place,
and $y =$ ... ... unit's ...

$\therefore$ $10x + y =$ the number; just in the same way as

$$23 = 10 \times 2 + 3,$$

and, $10y + x =$ the number with its digits inverted.

Hence the conditions of the question give us the following equations,

$$10x + y = 2(x + y) \dots\ (1.)$$

$$10x + y + 9 = \frac{10y + x}{3} \ \dots\ (2.)$$

From these equations we find $x = 1$, and $y = 8$, therefore the number required will be 18.

11. There is a number consisting of two digits, which is

equal to 12 times the difference of its digits; and if 36 be added to it, the digits will be inverted. Required the number. *Ans.* 48.

12. A and B commence trade with different sums of money; A gains £40, and then his capital is two times B's; A then loses £80, and B gains £70, and then B's capital is 6 times A's. How much had each at first?

Ans. £60 and £50.

13. Find two numbers whose sum is 2, and difference 1.

Ans. $1\frac{1}{2}$, and $\frac{1}{2}$.

41. When there are *three* equations given containing *three* unknown quantities, the values of these unknown quantities may be obtained after the same manner.

QUADRATIC EQUATIONS, WITH ONE UNKNOWN QUANTITY.

42. A quadratic equation is one which contains the unknown quantity in the second power or square. Equations of this kind are, in general, solved by the method of *completing the square*, which consists in adding a certain quantity to each side of the equation, so as to make the left hand member an exact square, or a quantity of which the square root can be exactly found: by this means the proposed equation is reduced to a simple one. The *form* which the square of a binomial assumes (see Art. **35.**) always enables us to complete the square of any quadratic expression. Thus, for example, as $(x+3)^2=x^2+6x+9$, if we had the expression, x^2+6x, it would be rendered a square quantity, by taking half the coefficient of x, that is the half of 6 or 3, and adding its square. In like manner, as $(x+5)^2 = x^2+10x+25$, if we had the expression, x^2+10x, it would be rendered a square quantity, by taking the half of 10 or 5, and adding its square. Again, as $(x-7)^2 = x^2-14x+49$, if we had the expression, x^2-14x, it would be rendered a square quantity, by taking the half of 14 or 7, and adding its square. And so on to other cases. After having com-

pleted the square, the square root is readily found by taking the square root of the first term and adding or subtracting (as the middle terms is + or −) the square root of the last term. Thus, as $x^2+8x+16=(x+4)^2$, the square root of this quantity will be $x+4$, where x is the square root of x^2, and 4 is the square root of 16. Again, as $x^2-6x+9=(x-3)^2$, the square root of this quantity will be $x-3$, where x is the square root of x^2, and 3 is the square root of 9.

Since $+3\times+3=9$, and $-3\times-3=9$, it follows that the square root of 9 may either be $+3$ or -3. This twofold nature of the root, in general, gives two values for the unknown quantity in a quadratic equation.

EXAMPLES IN QUADRATIC EQUATIONS.

1. Find the value of x in the equation,

$$x^2+6x=16.$$

Here to *complete the square*, we take the half of 6 which is 3, and then 3 squared gives us 9, which must be added to each side of the equation.

$$\therefore\ x^2+6x+9=16+9=25,$$
$$\text{or } (x+3)^2=25,$$

Taking the square root of each side,

$$x+3=5,$$
$$\therefore\ x=2.$$

This is one value of x, but another value may be found, because the root of 25 may be either -5 or $+5$; hence we have also,

$$x+3=-5,$$
$$\text{and } \therefore\ x=-8.$$

The values of x, therefore, are 2 or -8.

If either of these values be substituted for x in the given equation, the condition of equality will be satisfied.

2. Find the value of x in the equation,

$$x^2-8x=9.$$

Here the half of 8 is 4, and $4^2 = 16$, therefore to *complete the square*, we add 16 to each side of the equation,

$$\therefore\ x^2-8x+16 = 9+16 = 25,$$
$$\text{or } (x-4)^2 = 25.$$

Taking the square root of each side,

$$x-4 = 5 \text{ or } -5,$$
$$\therefore\ x = 9 \text{ or } -1.$$

3. $x^2+8x = 33$. *Ans.* $x = 3$ or -11.
4. $x^2+10x = 24$. ... $x = 2$ or -12.
5. $x^2+12x = 85$. ... $x = 5$ or -17.
6. $x^2-6x = 7$. ... $x = 7$ or -1.
7. $x^2-16x = 80$. ... $x = 20$ or -4.
8. $x^2-2x = 3$. ... $x = 3$ or -1.
9. Find the value of x in the equation,

$$x^2+5x = 14.$$

Here the half of 5 is $\frac{5}{2}$, which squared gives us $\frac{25}{4}$, therefore completing the square, we have,

$$x^2+5x+\frac{25}{4} = 14+\frac{25}{4} = \frac{81}{4}.$$

Taking the square root of each side of the equation,

$$x+\frac{5}{2} = \frac{9}{2} \text{ or } -\frac{9}{2},$$
$$\therefore\ x = 2 \text{ or } -7.$$

10. $x^2+3x = 40$. *Ans.* $x = 5$ or -8.
11. $x^2+7x = 30$. ... $x = 3$ or -10.
12. $x^2+9x = 10$. ... $x = 1$ or -10.
13. $x^2-7x = 8$. ... $x = 8$ or -1.
14. $x^2-13x = 140$. ... $x = 20$ or -7.
15. Find the value of x in the equation,

$$3x^2+5x = 22.$$

Before completing the square, we must always divide each side, by the coefficient of x^2, in order to get rid of that coefficient. Here dividing by 3, we find,

$$x^2+\frac{5}{3}x = \frac{22}{3}.$$

Here, in order to complete the square, we have the half of $\frac{5}{3}=\frac{5}{6}$, which squared gives, $\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$,

$$\therefore\ x^2+\frac{5}{3}x+\frac{25}{36}=\frac{22}{3}+\frac{25}{36}=\frac{289}{36},$$

Taking the square root of each side,

$$x+\frac{5}{6}=\frac{17}{6}\text{ or }-\frac{17}{6},$$

$$\therefore\ x=2\text{ or }-\frac{11}{3}.$$

16. $4x^2+6x=54$. *Ans.* $x=3$ or $-\frac{9}{2}$.
17. $8x^2-4x=24$. ... $x=2$ or $-1\frac{1}{2}$.
18. $9x^2-2x=7$. ... $x=1$ or $-\frac{7}{9}$.
19. Find the value of x in the equation,

$$x^2+\cdot8x=5\cdot6.$$

Completing the square by adding $\cdot4^2$ to each side, we have,

$$x^2+\cdot8x+\cdot16=5\cdot6+\cdot16=5\cdot76;$$

Taking the square root of each side,

$$x+\cdot4=\pm\sqrt{5\cdot76}=2\cdot4\text{ or }-2\cdot4,$$

$$\therefore\quad x=2\text{ or }-2\cdot8.$$

20. $x^2+\cdot26x=8\cdot3$. *Ans.* $x=2\cdot753$ or $-3\cdot013$.
21. $5x^2+3x=7\cdot25$. ... $x=\cdot9409$ or $-1\cdot5409$.
22. Find the value of x from the equation,

$$\frac{x+4}{x-4}+\frac{x-6}{2}=\frac{3x+4}{4}-3.$$

Here first multiplying each side of the equation by 4,

$$\frac{4x+16}{x-4}+2x-12=3x+4-12.$$

By transposing and collecting,

$$\frac{4x+16}{x-4}-x=4;$$

Multiplying each side by $x-4$,

$$4x+16-x^2+4x=4x-16;$$

Transposing and changing all the signs (Art. 8.);

$$x^2-4x=32;$$

Solving this quadratic, we find, $x=8$, or -4.

23. $5x = \frac{8-x}{x}+7.$ *Ans.* $x = 2$ or $-\frac{4}{5}$.

24. $\frac{x}{x-2} - \frac{x-1}{x} = 1+\frac{1}{x}.$ ... $x = 4$

25. $x^4+6x^2 = 40.$ ... $x = 2$ or -2.

Whenever the higher exponent of x is double that of the lower, the equation may always be solved by completing the square; thus,

$$x^4+6x^2+9 = 49;$$

Taking the square root,

$$x^2+3 = 7 \text{ or } -7,$$

$$\therefore\ x^2 = 4 \text{ or } -10;$$

Taking the square root,

$$x = \pm 2 \text{ or } \pm\sqrt{-10}.$$

The value $\pm\sqrt{-10}$ is called imaginary, because it cannot be actually calculated in numbers.

26. $x+8x^{\frac{1}{2}} = 20;$

Completing the square,

$$x+8x^{\frac{1}{2}}+16 = 36,$$

Taking the square root,

$$x^{\frac{1}{2}}+4 = 6,$$

$$\therefore x^{\frac{1}{2}} = 2.$$

Squaring each side, $x = 4.$

27. $3x^4+12x^2 = 351.$ *Ans.* $x = 3$ or -3.

28. $x^4-2x^2 = 224.$ ... $x = 4$ or -4.

29. $x+10x^{\frac{1}{2}} = 39.$ ... $x = 9$.

43. PROBLEMS PRODUCING QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

1. A farmer buys a certain number of sheep for 84*s.*; if he had bought 1 more for the same sum, each sheep would have cost 2*s.* less. How many did he buy?

Let $x =$ the number bought,

$\therefore\ \frac{84}{x} =$ price of each in shillings,

and $\frac{84}{x+1}$ = price of each had he bought 1 more.

But we are told that the latter price is 2*s.* less than the former,

$$\therefore \frac{84}{x+1}+2=\frac{84}{x}.$$

Multiplying first by x, and then by $x+1$, to get rid of the fractions, we have,

$$84x+2x^2+2x=84x+84,$$

$$\therefore x^2+x=42.$$

Solving this quadratic, we find $x=6$.

2. The product of two numbers is 15, and their difference is 2. Required the numbers. *Ans.* 3 and 5.

Let x = the less,

then $x+2$ = the greater,

$\therefore x\,(x+2)=15$,

or $x^2+2x=15$, whence $x=3$.

3. It is required to find a number, such that if it be increased by 12, the sum shall be equal to the square of the number. *Ans.* 4 or -3.

4. A person bought as many oranges as cost him 40 pence; after reserving 4, he sold the remainder for 48 pence, and thereby gained 1*d.* upon each. How many oranges did he buy? *Ans.* 20.

5. A draper bought a certain number of yards of cloth for £3 12*s.*, which he sold at 3*s.* 6*d.* per yard, and gained as much as 4 yards cost him. How many yards did he buy?

Ans. 24 yards.

6. A and B set out at the same time for a place at the distance of 20 miles; A travels 3 miles an hour faster than B, and completes the journey 6 hours before him. At what rate did each travel?

Let x = B's rate in miles an hour,

$\therefore x+3$ = A's ... ...

$\frac{20}{x}$ = No. hours in which B will complete the journey.

$\frac{20}{x+3}$ = ... A ... ...

But by the question, A completes the journey 6 hours sooner than B,

$$\therefore \frac{20}{x} - \frac{20}{x+3} = 6.$$

From this equation we find $x = 2$ miles which is B's rate per hour; and therefore A's rate per hour $= 5$ miles.

7. A person walked 21 miles, in 4 more hours than he travelled miles per hour. How many miles did he travel per hour? *Ans.* 3 miles.

8. A man walking at a certain rate, is able to complete a journey of 72 miles, 6 hours sooner than when he walks 1 mile per hour slower. At what rate per hour did he at first walk? *Ans.* 4 miles.

9. Divide 9 into two parts, so that the sum of their squares shall be equal to 53. *Ans.* 2 and 7.

10. There is a number consisting of two digits, the sum of which is equal to 7, and their product is 10 less than the number. Required the number. *Ans.* 16.

11. A and B can together complete a piece of work in a certain time; A alone can do it in 6 hours more, and B alone can do it in $1\frac{1}{2}$ hour more. In what time can A and B together do the work?

Let $x =$ the no. of hours, then

$\frac{1}{x} =$ fractional part of the work done by A and B in 1 hr.

$\frac{1}{x+6} =$ ... ... A alone ...

$\frac{1}{x+\frac{3}{2}} =$ ... ... B alone ...

But the sum of these two portions which A and B separately do in an hour, must be equal to the portion which they do in an hour when working together,

$$\therefore \frac{1}{x+6} + \frac{1}{x+\frac{3}{2}} = \frac{1}{x}.$$

Reducing this equation, we find $x^2 = 9$, and $\therefore x = 3$ hours.

12. A can do a certain work in 5 hours sooner than B, but they can together perform the work in 6 hours. In what time can each man do it? *Ans.* 10 and 15 hours.

13. A person travelled the distance of 40 miles; if he had gone 2 miles an hour faster than he did, he would have performed the journey 1 hour sooner. At what rate did he travel? *Ans.* 8 miles per hour.

14. I have a certain number of counters which I try to arrange in a square. At my first trial I have 5 counters over. I then enlarge the side of the square by one counter, and find that I have 2 counters too few to complete the square. How many counters have I?

Let $x =$ the no. of counters in the side of the 1st square. Now the question enables us to find the number of counters in two different ways, thus

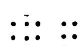
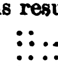
$$x^2+5 = \textit{no. counters},$$

$$\text{and } (x+1)^2-2 = \textit{no. counters},$$

$$\therefore\ (x+1)^2-2 = x^2+5.$$

Solving this equation we find $x = 3$, and $\therefore$ the number of counters $= 3^2+5 = 14$.

The following arrangement of counters verifies this result,

At the 1st trial  At the 2nd trial 

QUADRATIC EQUATIONS WITH TWO UNKNOWN QUANTITIES.

44. Many equations of this class may be solved by the artifice employed in Art. **39.**:

1. Find the values of x and y in the equations,

$$2x^2+xy = 14 \ \dots\ (1.)$$

$$3x + y = 9 \ \dots\ (2.)$$

From equation (2.) we obtain,

$$y = 9-3x.$$

Substituting this value of y in equation (1.), we have,

$$2x^2+x(9-3x) = 14,$$

$$\therefore\ x^2+9x = -14$$

Whence by solving this quadratic equation, we find $x = 7$ or 2; and substituting these two values for x, in the foregoing expression for y, we have $y = 9 - 3 \times 7 = -12$, or $y = 9 - 3 \times 2 = 3$. Where 7 and -12, 2 and 3 are the coincident values of x and y.

2. $\left.\begin{aligned} x^2 + xy &= 30 \\ 2x + y &= 11 \end{aligned}\right\};$ *Ans.* $x = 5$ or 6. ... $y = 1$ or -1.

3. $\left.\begin{aligned} 2x^2 - xy &= 12 \\ x - y &= 1 \end{aligned}\right\};$ ... $x = 3$ or -4. ... $y = 2$ or -5.

4. $\left.\begin{aligned} x^2 + y^2 &= 74 \\ x + 4y &= 33 \end{aligned}\right\};$ ... $x = 5$ or $-1\frac{2}{17}$. ... $y = 7$ or $8\frac{9}{17}$.

5. $\left.\begin{aligned} xy - y &= 9 \\ x + 2y &= 12 \end{aligned}\right\};$ ... $x = 10$ or 3. ... $y = 1$ or $4\frac{1}{2}$.

6. $\left.\begin{aligned} x^2 + y^2 + 4x &= 36 \\ \frac{x}{2} + 2y &= 6 \end{aligned}\right\};$ ... $x = 4$ or $-6\frac{6}{17}$. ... $y = 2$ or $4\frac{10}{17}$.

45. The following examples exhibit the use of various artifices in the solution of equations of this kind.

1. Find the values of x and y in the equations

$$x + y = 5 \ldots\ldots (1.)$$
$$xy = 6 \ldots\ldots (2.)$$

Squaring (1.) as a first step for finding the value of $x - y$,

$$x^2 + 2xy + y^2 = 25.$$

Multiplying equation (2.) by 4, we have

$$4xy = 24.$$

Subtracting this equation from the preceding one,

$$x^2 - 2xy + y^2 = 1,$$

Taking the sq. root, $x - y = 1$ or $-1 \ldots (3.)$

Adding this eq. and (1.), $2x = 6$ or 4.*

$$\therefore x = 3 \text{ or } 2.$$

Subtracting (3.) from (1.), $2y = 4$ or 6.

$$\therefore y = 2 \text{ or } 3.$$

2. $\left.\begin{aligned} x - y &= 1 \\ xy &= 20 \end{aligned}\right\};$ *Ans.* $x = 5$ or -4. ... $y = 4$ or -5.

* The student will now see the reason for seeking the value of $x - y$.

3. $$x-y=4\ldots(1.)$$
$$x^2+y^2=40\ldots(2.).$$

Squaring eq. (1.) as a first step for finding the value of $x+y$,
$$x^2-2xy+y^2=16,$$
Subtracting this eq. from (2.), $2xy=24$,
Adding this eq. to (2.), $x^2+2xy+y^2=64$,
Taking the sq. root, $x+y=8$ or $-8\ldots(3.)$
Adding this eq. to (1.), $2x=12$ or -4;
$\therefore x=6$ or -2.
Subtracting (1.) from (3.), $2y=4$ or -12;
$\therefore y=2$ or -6.

4. $\left.\begin{aligned} x+y &= 9 \\ x^2+y^2 &= 53 \end{aligned}\right\}$; *Ans.* $x=7$ or 2.
... $y=2$ or 7.

5. $\left.\begin{aligned} x^2+y^2 &= 17 \\ xy &= 4 \end{aligned}\right\}$; ... $x=1$ or 4.
... $y=4$ or 1.

6. $$x^2-y^2=24\ldots(1.)$$
$$x+y=6\ldots(2.)$$

By Art. **34.** eq. (1.) becomes,
$$(x+y)\,(x-y)=24.$$
Substituting the value of $x+y$ given in (2.),
$$6\,(x-y)=24,$$
Dividing by 6, $x-y=4$,
Adding this eq. to (2.), $2x=10$, and $x=5$.
Similarly subtracting, we find, $y=1$.

7. $\left.\begin{aligned} x^2-y^2 &= 12 \\ x-y &= 2 \end{aligned}\right\}$; *Ans.* $x=4$,
... $y=2$.

8. $x^2-y^2=32\ldots(1.)$ ... $x=6$ or -6.
$xy=12\ldots(2.)$ ... $y=2$ or -2.

Here, to the square of (1.), add 4 times the square of (2.), then take the square root, and add the resulting equation to (1.); then we have,
$$2x^2=72,\ \therefore x=\pm 6,\ \text{and}\ y=\pm 2.$$

9. $x+y=5\ldots(1.)$ *Ans.* $x=3$ or 2.
$x^3+y^3=35\ldots(2.)$ ... $y=2$ or 3.

From the cube of (1.) subtract (2.), then we have,
$$3x^2y+3xy^2=90.$$

Dividing by 3, and taking out the common factor,

$$xy\,(x+y) = 30.$$

Substituting (1.), $5xy = 30$, and $xy = 6$.

We have now, $x+y = 5$, and $xy = 6$; from these two equations we find the values of x and y, as in example 1.

10. $\left.\begin{aligned} x-y &= 1 \\ x^3-y^3 &= 7 \end{aligned}\right\};$ *Ans.* $x = 2$ or -1. ... $y = 1$ or -2.

11. $\left.\begin{aligned} x^2+y^2+xy &= 19 \\ x+y &= 5 \end{aligned}\right\};$ ... $x = 3$ or 2. ... $y = 2$ or 3.

12. $\left.\begin{aligned} x^2+xy &= 6 \\ y^2+xy &= 30 \end{aligned}\right\};$ ... $x = 1$ or -1. ... $y = 5$ or -5.

Add these equations together, and then take the square root, &c.

13. $\left.\begin{aligned} x^3+y^3 &= 9 \\ x^2y+xy^2 &= 6 \end{aligned}\right\};$ *Ans.* $x = 2$ or 1 ... $y = 1$ or 2.

Take three times the second, and add the result to the first; then take the cube root, &c.

14. $$x^2+y^2+x+y = 32 \;\ldots\; (1.)$$
$$xy = 12 \;\ldots\; (2.)$$

Take 2 times (2.), and add to (1.), then,

$$(x+y)^2 + (x+y) = 56.$$

This equation may be solved as a quadratic for $x+y$; hence we have, completing the square (by adding $\frac{1}{4}$), and then taking the square root,

$$x+y+\tfrac{1}{2} = \frac{15}{2}.$$

$$\therefore x+y = 7.$$

This equation taken with (2.), will enable us to find x and y, as in example 1. Hence we find $x = 4$ or 3, and $y = 3$ or 4. Other roots may be found by taking the minus sign of the square root.

15. $\left.\begin{aligned} x^2+y^2+x-y &= 6. \\ xy &= 2. \end{aligned}\right\};$ *Ans.* $x = 2$ or -1. ... $y = 1$ or -2.

16. $$x-y = 2 \;\ldots\; (1.)$$
$$x^4+y^4 = 272 \;\ldots\; (2.)$$

Here the unknown quantities may be found, by the same

method as that given in example 9; but it will also be instructive to attend to the following method:—

Putting $x = u+v$, and $y = u-v$, then (1.) becomes,

$$2v = 2, \text{ or } v = 1;$$

and (2.) becomes, after a little reduction,

$$u^4+6u^2v^2+v^4 = 136.$$

Substituting 1 for v (as found above), we have,

$$u^4+6u^2 = 135.$$

Solving this quadratic, by the method explained in Example 25. Art. **42.**, we find $u = 3$, or -3; and

$\therefore x = u+v = \pm 3+1 = 4$ or -2; and $y = 2$ or -4.

The artifice here exhibited may be used with advantage whenever x and y are similarly involved.

17. $\left.\begin{aligned} x+y &= 4 \\ x^4+y^4 &= 82 \end{aligned}\right\}$; *Ans.* $x = 3$ or 1. $y = 1$ or 3.

46. When the equations are homogeneous, that is, when every unknown term is of two dimensions, they may *always* be solved by the following method:—

1. $x^2+xy = 21 \ldots (1.)$
   $y^2-xy = 4 \ldots (2.)$.

Since x must be some ratio of y, we may put $z\,y$ for x. Making this substitution, the two equations become,

$$z^2y^2+zy^2 = 21; \therefore y^2 = \frac{21}{z^2+z}.$$

$$y^2-zy^2 = 4; \therefore y^2 = \frac{4}{1-z} \ldots (3.).$$

Therefore, by equating the values of y^2,

$$\frac{4}{1-z} = \frac{21}{z^2+z}.$$

Solving this quadratic equation, we find $z = \frac{3}{4}$. Now substituting $\frac{3}{4}$ for z in eq. (3.) we find $y^2 = 16$; and $\therefore y = \pm 4$. But $x = zy = \frac{3}{4} \times \pm 4 = \pm 3$.

Other values may be found by taking the minus value of z.

2. $\left.\begin{aligned} 2x^2-3xy &= 8 \\ 4y^2+3x^2 &= 64 \end{aligned}\right\}$; *Ans.* $x = 4$ or -4. $y = 2$ or -2.

3. $\left.\begin{aligned} x^2+2y^2 &= 57 \\ 3x^2+xy &= 95 \end{aligned}\right\};$ *Ans.* $x = 5$ or $-6{\cdot}164$.
... $y = 4$ or $-3{\cdot}082$.

4. $\left.\begin{aligned} 2x^2-y^2 &= 14 \\ 2y^2-x^2 &= -1 \end{aligned}\right\};$ ... $x = 3$ or -3.
... $y = 2$ or -2.

47. PROBLEMS IN SIMULTANEOUS EQUATIONS OF TWO UNKNOWN QUANTITIES.

1. A person bought a certain number of yards of cloth, for a certain sum; if he had bought 1 yard more than he did for the same money, he would have paid 2*s.* per yard less: and if he had bought 1 yard less for the same money, he would have paid 4*s.* per yard more. How many yards did he buy, and how much did he pay for each?

Let $x =$ no. of yards,
and $y =$ cost of each yard in shillings,
then $xy =$ *the total cost in shillings.*

But, from the question, we can also obtain two other independent values for the total cost, viz.

$(x+1)(y-2) =$ *the total cost in shillings,*
and $(x-1)(y+4) =$ *the total cost in shillings,*

$$\therefore (x+1)(y-2) = xy \quad \ldots \quad (1.)$$
$$\text{and, } (x-1)(y+4) = xy \quad \ldots \quad (2.)$$

Performing the multiplications in these two equations, and transposing, we find the two following equations,

$$y-2x = 2,$$
$$\text{and } 4x-y = 4.$$
$$\text{Adding,} \quad 2x = 6,$$
$$\therefore \quad x = 3, \text{ the no. yds.}$$

Substituting this value of x, we find $y = 8$ shillings.

2. The product of two numbers is 6, and the sum of their squares is 13; what are the numbers? *Ans.* 2 and 3.

3. What number is that, which being divided by the sum of its two digits, the quotient is 4; and the product of the digits is equal to one half the number? *Ans.* 36.

4. If the numerator of a certain fraction be multiplied by

its denominator, the product will be 12; and if 5 be added to the numerator, the fraction will be equal to 2. Required the fraction. *Ans.* $\frac{3}{4}$.

5. A man performed a certain journey; had he gone 1 mile an hour quicker than he did, he would have completed the journey in three-fourths of the time; and if had gone 1 mile an hour slower than he did, he would have been 5 hours longer in the journey. Required the distance, and the rate of travelling.

Ans. Distance $=30$ miles, and rate per hour $=3$ miles.

6. A grocer sold 8 lbs. of sugar, and 10 lbs. of coffee for 19*s.*; but it took 4 lbs. more of sugar than it took of tea to come to 3*s.* What is the cost of each per lb.?

Ans. 6*d.* and 18*d.*

Let $x=$ the cost of the sugar per lb. in shillings,

and $y=$ ... tea ... ...

then, $\frac{3}{x}=$ no. lbs. of sugar to cost 3*s.*

and, $\frac{3}{y}=$ ... tea ...

But the former is 4 greater than the latter,

$$\therefore \quad \frac{3}{x}-\frac{3}{y}=4 \quad \ldots \quad (1.)$$

But, moreover, we have for the whole cost,

$$8x+10y=19 \quad \ldots \quad (2.)$$

By solving these equations, the values of x and y are found.

7. Three lambs and one sheep were sold for 36*s.*; but it takes 2 more lambs than it takes sheep to cost 48*s.* Required the price of each. *Ans.* 12*s.* and 8*s.*

LITERAL EQUATIONS.

48. In the preceding examples of equations, the known quantities, or data, are expressed in numbers, and hence the answer, when found, does not exhibit the *process* whereby it has been deduced. Now, when letters, as a, b, c, &c., are

put for the given things in a problem, the final result shows, how the answer to any particular case of the problem may be obtained from the given numbers.

Let us work out example 3. Art. 5. with general quantities for the data.

1. A farmer bought a cow and a horse for a £; now the horse cost b £ more than the cow. Required expressions for the cost of each.

Let x = cost of the cow in pounds,

$\therefore\ x+b =$ ... horse ...

$$\therefore\ 2x+b = a.$$

Taking b from each side of the equation,

$$2x = a-b,$$

Dividing by 2, $x = \dfrac{a-b}{2}$.

But the cow and the horse together cost a £.

$$\therefore\ \text{Cost of the horse} = a-\frac{a-b}{2} = \frac{a+b}{2}.$$

If, as in example 3. Art. 5., $a = 13$, and $b = 3$, then,

$$x = \frac{13-3}{2} = £5.$$

2. A servant contracted for c £ a year and his livery. At the end of m months he was turned away, and then received p £ and his livery. Required an expression for the value of the livery. (See Example 6. Art. 24.)

Let x = the value of the livery in pounds,

then wages for 12 months $= c+x$,

$\therefore$ ... 1 month $= \dfrac{c+x}{12}$,

$\therefore$ ... m months $= m$ times $\dfrac{c+x}{12} = \dfrac{cm+mx}{12}$.

But the wages received for m months $= p+x$.

$$\therefore\ p+x = \frac{cm+mx}{12},$$

Multiplying each side of the equation by 12,

$$12p+12x = cm+mx,$$

Transposing, $12x-mx = cm-12p$

Collecting the x's, see Art. **12.**,

$$x(12-m) = cm-12p$$

Dividing by $12-m$, $x = \dfrac{cm-12p}{12-m}$.

If (as in the example referred to) $c=9$, $m=5$, and $p=2$,

$$\text{then } x = \frac{9\times5-12\times2}{12-5} = £3.$$

3. A farmer bought a certain number of sheep; if he had paid a shillings a head more, they would have cost b shillings; but if he had paid c shillings a head less, they would have cost d shillings. Required an expression for the cost of each. (See Example 6. Art. **26.**)

Let $x =$ the no. shillings each sheep cost;

then, $\dfrac{b}{x+a} =$ *no. sheep*,

and also, $\dfrac{d}{x-c} =$ *no. sheep*;

$$\therefore \frac{b}{x+a} = \frac{d}{x-c}.$$

Clearing this equation of fractions, we have,

$$bx-bc = dx+ad,$$

Transposing, $bx-dx = bc+ad$,

Collecting the x's, $x(b-d) = bc+ad$,

Dividing by $b-d$, $x = \dfrac{bc+ad}{b-d.}$

If $a=2$, $b=60$, $c=3$, and $d=35$, then

$$x = \frac{60\times3+2\times35}{60-35} = 10s.$$

4. The sum of two numbers is s, and the difference d; required an expression for the greater. *Ans.* $\dfrac{s+d}{2}$.

5. If a be added to a number it will make it b times greater. What is the number? *Ans.* $\dfrac{a}{b-1}$.

6. A person distributed a shillings among b people; to

one portion he gave c shillings a piece, and to the rest d shillings a piece. How many people were there of each class? *Ans.* $\dfrac{a-bd}{c-d}$ and $\dfrac{bc-a}{c-d}$.

7. A cistern is supplied with water by a pipe; after the pipe had been running for t minutes, there were g gallons of water in the cistern; and after it had been running for $t_{,}$ minutes, there were $g_{,}$ gallons in the cistern. Required the number of gallons which run out of the pipe per minute.

Let $x =$ no. gals. run out in 1 minute,

then $tx =$ ... ... t minutes,

and $t_{,}x =$ ... ... $t_{,}$ minutes.

$$\therefore\ t_{,}x - tx = g_{,} - g,$$

$$\therefore\ x = \frac{g_{,}-g}{t_{,}-t}.$$

8. A could reap a field in a days, but if B assisted him for b days, he could reap it in c days. In how many days could B alone finish it?

Let $x =$ no. days in which B could finish it, then supposing the whole work to be represented by unity, we have,

$\dfrac{1}{x} =$ part done by B in 1 day,

$\therefore\ \dfrac{b}{x} =$ ... ... B in b days,

Moreover $\dfrac{1}{a} =$ ... ... A in 1 day,

$\therefore\ \dfrac{c}{a} =$ ... ... A in c days.

But the part done by B for the time he works, added to the part done by A for the time he works, must make up the whole work, or unity.

$$\therefore\ \frac{b}{x} + \frac{c}{a} = 1,$$

$$\therefore\ \frac{b}{x} = 1 - \frac{c}{a} = \frac{a-c}{a},$$

$$\therefore\ x = \frac{ab}{a-c}.$$

9. An island is a miles in circumference; A starts to walk round it at the rate of b miles per hour; c hours after B starts to walk, in the same direction, at the rate of d miles per hour; in what time will B overtake A? *Ans.* $\frac{bc}{d-b}$.

10. Two couriers, travelling at the rates of m and $m_{,}$ miles per hour, pass through a place at an interval of h hours. At what distance will the second courier overtake the first? *Ans.* $\frac{m_{,}mh}{m_{,}-m}$.

11. A waterman rows a miles per hour with the tide, and b miles per hour against the tide; required the rate of the tide. *Ans.* $\frac{a-b}{2}$.

RATIOS AND PROPORTION.

49. The ratio of two numbers is their relative magnitude, or the number of times that the one number is greater than the other, or, what is the same thing, the number of times that the one number is contained in the other; thus the ratio of a to b means the number of times that a is greater than b, and the ratio is therefore expressed by $\frac{a}{b}$; for example, if a represent 5, and b represent 4, then the ratio of 5 to 4 is expressed by saying, that 5 are 5 times the 4th of four, that is, the ratio is $\frac{5}{4}$.

Every question in the "Rule of Three" may be wrought out on the principle of ratios, as in the following problems.

Problem 1. If a articles cost £b, what will c articles of the same sort cost?

Here, cost a articles $=$ £b.

$\therefore$ ... 1 ... $=$ £$\frac{b}{a}$.

$\therefore$ ... c ... $= c$ times £$\frac{b}{a}$. $= \frac{bc}{a}$£.

In order to render the foregoing solution as simple as possible, let us take the following example.

Example. If 5 articles cost £2, what will 4 articles cost?

Here, cost 5 articles = £2.

$\therefore$ ... 1 ... = £$\frac{2}{5}$.

$\therefore$ ... 4 ... = 4 times £$\frac{2}{5}$ = £ $\frac{8}{5}$.

In the course of the demonstration, the following questions may be put by the teacher.

Teacher. (Writing down, "*cost* 5 *articles* ="). What is the cost of 5 articles?

Pupil. Two pounds.

Teacher. (Writing down, "*cost* 1 *article* = "). Will the cost of 1 article be more or less than £2?

Pupil. It will be less.

Teacher. How many times less?

Pupil. Five times less, that is $\frac{1}{5}$ of £2.

Teacher. But we have not to find the cost of one article, — how many have we to find the cost of?

Pupil. Four articles.

Teacher. Will the cost of 4 articles be more or less than the cost of one?

Pupil. More; it will be 4 times more.

Problem 2. If a men can do a certain piece of work in b days, how many men will it take to do the same work in c days?

Here, no. men to do the work in 1 day = b times $a = ab$.

$\therefore$ .. .. c days $= \frac{ab}{c}$.

50. When four terms have an equality of ratio, these terms form a proportion; thus, if $\frac{a}{b}=\frac{c}{d}$, we have the proportion $a : b :: c : d$, by which therefore, is simply meant, that the first term is as many times as the second, as the third is that

of the fourth. From this definition of a proportion, the following properties, or theorems, may easily be deduced.

1st. If $a : b :: c : d$,

$$\text{then } \frac{a}{b} = \frac{c}{d},$$

$$\therefore ad = bc.$$

That is, in any proportion, *the product of the extremes is equal to the product of the means.*

2nd. If $a : b :: c : d$, and $a : b :: e : f$,

then, $c : d :: e : f$;

because from the first proportion, $\frac{a}{b} = \frac{c}{d}$,

and from the second ... $\frac{a}{b} = \frac{e}{f}$;

$\therefore \frac{c}{d} = \frac{e}{f}$, then, putting this equality into the form of a proportion, we find the property above enunciated.

3d. If $a : b :: c : d$, then $a + b : b :: c + d : d$.

For since $\frac{a}{b} = \frac{c}{d}$, by adding one to each side, we have,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1.$$

Bringing the quantities to the same denominator, and adding, we have,

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Putting this equality of ratio into a proportion,

$$a + b : b :: c + d : d.$$

By similar artifices, various other properties may be readily deduced.

PROBLEMS REQUIRING THE USE OF PROPORTION.

1. Divide £64. between two persons, so that their shares may be to each other as 9 is to 7.

Let $x =$ the 1st person's share,

then $64 - x =$ ... 2nd ...

Then we have by the given ratio,

$$x : 64 - x :: 9 : 7,$$

Taking the product of the extremes and means, by theorem 1st, we have,

$7x = (64 - x)\ 9$, solving this equation we find,

$x =$ £36. the 1st person's share.

And the 2nd person's share $= 64 - 36 =$ £28.

2. Divide 20 into two parts, which shall be to each other as 2 is to 3. *Ans.* 8 and 12.

3. It is required to find two numbers, such that their product shall be equal to 24 times their difference; and that the greater shall be to the less as 4 is to 3. *Ans.* 8 and 6.

4. A sets out from Alnwick to Durham, at the same time that D sets out from Durham to Alnwick; A arrives at Durham 4 hours, and D at Alnwick 9 hours after they met. In what time did each man complete the journey?

Ans. A's time = 10 hours, and B's time = 15 hours.

5. A line of 20 inches is divided into two parts, whose difference is 10; required the proportion of the parts.

Ans. 1 to 3.

ARITHMETICAL SERIES.

51. An arithmetical series is a collection of quantities, constantly increasing, or decreasing, from one to the other by a certain number. Thus, $3+5+7+9+11$, is an arithmetical series, where the number of terms is 5, the first term 3, the last term 11, and the common difference 2. Here as any term is 2 greater than the term which goes before it, we may write the series in the following form;

1st term $= 3$
2nd term $= 3+2$
3rd term $= 3+2$ times 2
4th term $= 3+3$ times 2:

where the number of 2's in any term is one less than the place of that term.

In general let a be the first term, and d the common difference, then,

1st term $= a$

$$\begin{aligned}
&\text{2nd term} = a+d\\
&\text{3rd term} = a+2d\\
&\text{4th term} = a+3d\\
&\quad\vdots \qquad\quad \vdots\\
&n\text{th term} = a+(n-1)\,d;
\end{aligned}$$

where the number of d's in any term is one less than the place of that term.

Knowing the way in which the terms of a series are formed, we shall be able to find any term without writing down, as we might in some cases do, all the terms which go before. Thus let it be required to find the 6th term of the series, $2+6+10+14+$ &c. Here as the first term is 2, and the common difference 4, we have, the 6th term $= 2+5$ times $4 = 22$; the factor 5 being one less than the number of the term.

EXAMPLES.

1. Required the 7th term of the series, $1 + 5 + 9 + 13 +$ &c.

Here the 1st term $= 1$, and the common difference $= 4$,

$\therefore$ the 7th term $= 1+6\times 4 = 25$.

2. Required the 10th term of the preceding series *Ans.* 37.

3. Required the 41st term of the series, $7+8+9+$ &c. *Ans.* 47.

4. Required the 11th term, and the term preceding it, of the series, $3+8+13+18+$ &c. *Ans.* 53 and 48.

The sum of a series, s, is the addition, or collection, of all the terms composing it.

Let it be required to find the sum of the following series,

$$s = 1+3+5+7+9+11+13.$$

Here writing the series in an inverted form,

$$s = 13+11+9+7+5+3+1,$$

Adding these series together, we have,

$$2s = 14+14+14+14+14+14+14.$$

Here the number of times that 14 is taken is 7, which is the number of terms in the given series.

$$\therefore\ 2s = 7 \times 14\ ;\ \therefore\ s = \frac{7 \times 14}{2} = 49.$$

This result shows that the *sum of the series is equal to the sum of the first and last terms multiplied by half the number of terms.*

In general let it be required to find the sum of the series,

$$s = a + (a+d) + (a+2d) + \ldots \text{ to } n \text{ terms.}$$

Let l be put for the last, or nth term, then the term going before this will be $l-d$, and the term before this latter will be $l-2d$, and so on. Hence the proposed series may be written,

$$s = a + (a+d) + (a+2d) + \ \ldots\ + (l-2d) + (l-d) + l,$$

writing the terms of the series in an inverse order,

$$s = l + (l-d) + (l-2d) + \ \ldots\ + (a+2d) + (a+d) + a.$$

Adding the corresponding terms together, we find that the d's in each term destroy each other, and

$$\therefore 2s = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l).$$

But as $a+l$ is here repeated n times, we have,

$$2s = n \text{ times } (a+l)\ ;\ \therefore\ s = (a+l)\frac{n}{2} \ldots (1.)$$

But, l, or the nth term $= a + (n-1)d$, hence we have, by substituting this value for l,

$$s = (a + a + \overline{n-1}d)\frac{n}{2} = (2a + \overline{n-1}d)\frac{n}{2} \ldots (2.)$$

EXAMPLES.

1. Find the sum of the series, $1+3+5+$ &c. to 20 terms.

Here, the 20th term $= 1 + 19$ times $2 = 39$; hence the series may be written,

$$s = 1 + 3 + 5 + \ \ldots\ + 35 + 37 + 39,$$

writing the terms in an inverse order,

$$s = 39 + 37 + 35 + \ \ldots\ + 5 + 3 + 1.$$

Adding the corresponding terms together,

$$2s = 40 + 40 + 40 + \ \ldots\ + 40 + 40 + 40.$$

But we have 20 terms in this series,

$$\therefore\ 2s = 20 \text{ times } 40 = 800$$

$$\therefore\ s = 400, \text{ the sum of the series.}$$

Or thus by the formula,

Here, $a=1$, $d=2$, and $n=20$, then by substituting these values, we have

$$s = (2a+\overline{n-1}d)\frac{n}{2} = (2\times 1+19\times 2)\frac{20}{2} = 400.$$

2. Find the sum of $1+\ 4+\ 7+$&c. to 18 terms. *Ans.* 477.
3. Find the sum of $5+\ 9+13+$&c. to 30 terms. *Ans.* 1890.
4. Find the sum of $1+10+19+$&c. to n terms.

Ans. $\dfrac{9n^2-7n}{2}$.

5. The first term of an arithmetical series is 2, the common difference 3, and the sum of the series 40; required the number of terms.

Let $x =$ the number of terms,

then by the formula (2.), we have by putting,

$$a=2,\ d=3,\ s=40, \text{ and } x \text{ for } n,$$

$$(4+\overline{x-1}\times 3)\frac{x}{2} = 40,$$

By reducing this equation, we have,

$$3x^2+x = 80,$$

Solving this quadratic, we find, $x = 5$, the number of terms.

6. The first term of an arithmetical series is 1, the common difference 1, and the sum of the series 36; required the number of terms. *Ans.* 8.

7. A regiment of soldiers is drawn up in the shape of a solid equal-sided wedge. Find the number of soldiers, when the outer rank contains 36 men. *Ans.* 666.

GEOMETRICAL SERIES.

52. A geometrical series is formed, when any one of the terms is obtained from its preceding term by multiplying by a certain number called the common ratio. Thus, $3+6+12$

$+24+$&c., is a geometrical series, where the 2nd term is obtained from the first, by multiplying by 2; the 3rd term from the second by multiplying by 2; and so on. In this case the common ratio is 2.

Having given the first term, the common ratio, and the number of terms, the series may be written down. Thus, if the first term be 2, the common ratio 3, and the number of terms 4, we shall have the series $2+2.\ 3+2.\ 3^2+2.\ 3^3$, where the exponent of the 3, in any term, is one less than its place in the series; thus, in the third term, this exponent is 2; in the 4th term it is 3; and therefore in the nth term it will be $n-1$.

Let it be required to find a short expression for the sum of the following geometrical series.

$$s=1+2+4+8+16+32+64.$$

Multiplying each side by 2, the common ratio, we have,

$$2s=2+4+8+16+32+64+128.$$

Subtracting the first series from the second, we have,

$$s=128-1,$$

The sum of the series, in this case, is simply the difference between the first and twice the last term.

Every repeating decimal is a geometrical series; for example, $\cdot 3333'$ is a geometrical series having $\frac{1}{10}$ for the common ratio, and where the terms are continued without end, or to infinity. Let it be required to find the value of the repeating decimal, $\cdot 555'$.

Here, $s=\cdot 5555$ to infinity.

Multiplying by 10, observing that as the terms are continued to infinity, there will be 5 carried from every product,—

$$\therefore\ 10s=5\cdot 5555 \text{ to infinity.}$$

Subtracting the given series from this, we have,

$$9s=5, \text{ and } \therefore\ s=\tfrac{5}{9}.$$

Again, let us take the infinite series,

$$s=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\&c \text{ to infinity.}$$

Multiplying by $\frac{1}{2}$, the common ratio,

$$\tfrac{1}{2}s=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\&c. \text{ to infinity.}$$

Subtracting the latter from the former,

$$\tfrac{1}{2}s = 1; \text{ and } \therefore s = 2.$$

Let us now take the general form of the series, and for this purpose, let a be the first term, r the common ratio, and n the number of terms, then,

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}.$$

Multiplying each side by r, we have,

$$rs = ar + ar^2 + \ldots + ar^{n-1} + ar^n,$$

Subtracting the first equality from the second,

$$rs - s = ar^n - a; \ \therefore s(r-1) = a(r^n - 1);$$

$$\therefore s = \frac{a(r^n - 1)}{r-1} \ldots (1.)$$

When r is a fraction rs will be less than s, and then we must subtract the second series from the first; in this case we find,

$$s = \frac{a(1-r^n)}{1-r} \ldots (2.)$$

If the series be continued to infinity, then n will be infinite, and the quantity r^n will become nothing; for a fraction multiplied by itself, gets less and less as we proceed with the multiplication. In this case, therefore, eq. (2.), becomes $s = \dfrac{a}{1-r} \ldots (3.)$, which is the sum of a geometrical series continued to infinity.

EXAMPLES.

1. Find the sum of $1+4+16+$&c. to 7 terms. *Ans.* 5461.
2. ... ... of $1+3+9+$&c. to 8 ... ... 3280.
3. ... ... of $1+10+100+$&c. to 5 terms ... 11111.
4. ... ... of $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$&c. to infinity. ... $\frac{1}{2}$.
5. ... ... of $1+\frac{1}{4}+\frac{1}{16}+$&c. to infinity. ... $\frac{4}{3}$.

6. Required the value of the repeating decimal ·3737′, where the recurring figures are 37.

Here, $s = \cdot 3737$ to infinity.

Since the common ratio in this case is $\frac{1}{100}$, we multiply each side by 100.

$$\therefore \; 100s = 37{\cdot}37 \text{ to infinity}$$

$$\text{Subtracting, } 99s = 37\text{; and } \therefore \; s = \tfrac{37}{99}.$$

From this result we derive the following rule for finding the value of a recurring decimal: *Write the repeating figures, as the numerator of a fraction, and place the same number of nines below them, as the denominator of the fraction.*

7. What is the value of the repeating decimal $\cdot 2525'$?

Ans. $\frac{25}{99}$.

8. What is the sum of the series, $2+2^2+ \ldots +2^n$?

Ans. $2^{n+1}-2$.

9. ... ... , $a+a^2+ \ldots +a^n$?

Ans. $\dfrac{a^{n+1}-a}{a-1}$.

10. Let $s = 1+(1+r)+(1+r)^2+ \ldots (1+r)^{n-1}$;

$\therefore \; (1+r)\,s = (1+r)+(1+r)^2+ \ldots (1+r)^{n-1}+(1+r)^n$,

$$\therefore \; rs = (1+r)^n-1\text{; and } \therefore \; s = \frac{(1+r)^n-1}{r}.$$

53. The sum of infinite series, like the following, may also be readily found.*

Ex. 1. Find the sum of the infinite series,

$$s = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c.$$

Here every term of the series may be decomposed into two simple fractions; thus, for example,

$\dfrac{1}{3.4} = \dfrac{1}{3} - \dfrac{1}{4}$, and so on to other cases. Hence the given series may be written,

$$s = (\tfrac{1}{1}-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+\tfrac{1}{3}-\tfrac{1}{4})+\&c.$$

$\therefore \; s = 1$; for the plus quantity in any term destroys the minus quantity in the preceding term.

* See Tate's Factorial Analysis, with the Summation of Series.

2. Find the sum of the infinite series,

$$s = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \&c. \qquad \textit{Ans. } s = \tfrac{1}{2}$$

SIMPLE INTEREST.

54. Interest is the sum paid for the loan of money.

PROB. Given £P the principal or money lent; £r. the interest of £100. for 1 year; n the number of years that the principal remains at interest; it is required to determine an expression for I the interest for this time.

Here, Interest for £100. for 1 year $= r$

$$\therefore \quad \ldots \quad £1. \quad \ldots \quad = \frac{r}{100}$$

$$\therefore \quad \ldots \quad £P. \quad \ldots \quad = P \text{ times } \frac{r}{100} = \frac{rP}{100}$$

$$\therefore \quad \ldots \quad £P. \text{ for } n \text{ years} = n \text{ times } \frac{rP}{100} = \frac{nrP}{100}.$$

$$\text{That is, } I = \frac{nrP}{100} \;\ldots\; (1.).$$

From this formula we deduce the following rule :

To find the interest of any sum, we multiply the principal, the rate per cent., and the number of years together ; and then divide this product by 100.

Let M be put for the *amount*, or the sum of the principal and interest,

$$\text{then, } M = P + I,$$

Substituting the value of I given in (1),

$$M = P + \frac{nrP}{100} = P\left(1 + \frac{nr}{100}\right) \;\ldots\; (2.).$$

Any three of the four general terms, in formulas (1.) and, (2.), being given, the remaining one may be found by solving the resulting equation.

Ex. 1. In how many years will the simple interest of £40 at 5 per cent., amount to £6?

Here, $P = 40$, $r = 5$, and $I = 6$, then by (1.) we have,

$$\frac{n \times 5 \times 40}{100} = 6 \text{; and } \therefore n = 3 \text{ years.}$$

2. What must be the rate per cent., so that £50 may produce £5 interest in $2\frac{1}{2}$ years? *Ans.* 4.

3. What sum or principal will amount to £40 in 3 years at 5 per cent. per annum simple interest?

Here by formula (2.) we have $M = 40$, $n = 3$, and $r = 5$, to find P,

$$\therefore P\left(1 + \frac{3 \times 5}{100}\right) = 40$$

$$\therefore P = £34\ 15s.\ 7\tfrac{19}{23}d.$$

DISCOUNT.

55. Discount is the allowance which must be made for the *present payment* of a debt, which is not due until a certain time hence. Suppose that I am owing a person £104, which I have to pay 1 year after this time; and let us also suppose that £100 bring £4 interest in 1 year; then, in fairness, I ought to pay him £100 at the present time; for he, putting this £100 out at interest, receives £4 at the end of 1 year, so that he then will just receive the money I had to pay him. In this case the discount is £4, and the present payment £100.

PROB. Let M £ be the money to be paid at the end of n years; and r £ the interest of £100 in 1 year; required an expression for D, the discount.

Here, interest of £100 for 1 year $= r$

$\therefore$ ... ... n years $= nr$

Hence the present payment of $(100+nr)$ £, will be £100,

$\therefore$ Discount upon $(100+nr)$ £ $= nr$

$$\therefore \quad \ldots \quad \ldots \quad 1\ £ = \frac{nr}{100+nr}$$

$$\therefore \quad \ldots \quad \ldots \quad M\ £ = \frac{nrM}{100+nr}$$

That is, $D = \dfrac{nrM}{100+nr}$ ... (1.)

Any three of the four general quantities in this formula being given, the remaining one may be found, by solving the resulting equation.

To find the present payment, we have only to subtract the discount from the debt, or to take D from M.

Ex. 1. I had to pay a certain sum of money 1 year hence, but upon paying the money at the present time, I received £3 discount, required the debt, supposing the interest of money to be worth 4 per cent.

Here, $D = 3$, $n = 1$, $r = 4$, and M is required,

$$\therefore \quad \frac{1 \times 4 \times M}{100 + 1 \times 4} = 3 \,; \ \therefore \ M = £78.$$

2. The discount upon a bill of £530, payable at 2 years hence, was £30; required the rate of interest.

Ans. 3 per cent.

LOSS AND GAIN.

56. The loss or gain upon the sale of any goods, is usually calculated at a certain rate per cent. upon the cost price of the goods. In all questions of this sort we always have either of the two following relations;

Selling price = Cost price + Gain

Selling price = Cost price − Loss.

Prob. Let C £ be the cost price of the goods, S £ the selling price, and r the gain per cent. upon the cost price; required an expression for S.

Here, Gain upon £100 $= r$

$$\therefore \quad \ldots \quad \ldots \quad £1 = \frac{r}{100}$$

$$\therefore \quad \ldots \quad \ldots \quad C\,£ = C \text{ times } \frac{r}{100} = \frac{rC}{100}$$

$$\therefore S = C + \frac{C}{100} = C \times \frac{100 + r}{100} \ldots (1.)$$

In like manner, if there be r per cent. lost, we find,

$$S = C \times \frac{100-r}{100} \ldots (2.)$$

or, expressing these results in one formula,

$$S = C \times \frac{100 \pm r}{100} \ldots (3.)$$

Where the + sign is used when there is *gain*, and the − sign when there is *loss*.

Any three of these general quantities being given, the remaining one may found.

Ex. 1. A merchant bought goods for £30, and sold them for £57; how much did he gain per cent.?

Here $C = 30$, $S = 57$, and r is required, then by formula (1.),

$$30 \times \frac{100+r}{100} = 57\text{; and } \therefore r = 90 \text{ per cent.}$$

2. A person sold goods for £53, and thereby gained 6 per cent.; required the cost price of the goods.

Let x = cost price in pounds,

then gain upon £1 $= \frac{6}{100}$

$$\therefore \quad \ldots \quad x£ = x \text{ times } \frac{6}{100} = \frac{6x}{100};$$

But, Cost price + Gain = Selling price,

$$\therefore x + \frac{6x}{100} = 53,$$

$$\therefore x = £50.$$

3. Sold goods for £46, and thereby lost at the rate of 8 per cent. Required the cost price. *Ans.* £50.

4. A merchant sold goods for £39, and gained as much per cent. as they cost him. Required the cost price of the goods.

Let x = cost price in pounds.

But, by the question, we have

Gain upon £100 $= x$

$$\therefore \text{ Gain upon } £1 = \frac{x}{100}.$$

$$\therefore \quad \ldots \quad \ldots \quad £x = x \text{ times } \frac{x}{100} = \frac{x^2}{100}$$

$$\therefore \ x + \frac{x^2}{100} = 39.$$

Solving this quadratic equation, we find, $x = £30$.

5. A person sold a horse for £12, and gained twice as much per cent. as it cost him. What did he pay for the horse?

Ans. £10.

COMPOUND INTEREST.

57. When a person receives compound interest upon a sum of money, the interest, as it becomes due, is put to the principal, and then the succeeding interest is chargeable on this sum, so that the principal at the end of any year is the principal of the preceding year added to the interest arising during that year.

Prob. To find the amount of P£ at compound interest for n years, the interest being payable yearly.

Let $R = \frac{r}{100}$, the interest of £1 for one year; and let M_1, M_2, M_3, ... M_n, be put for the amounts at the end of the 1st, 2nd, 3rd, ... and nth years, then,

$$\text{Amount} = \text{principal} + \text{interest},$$

$$\therefore \ M_1 = P + PR = P(1+R).$$

In like manner, if Q be the amount, at the end of any year, then, Q $(1+R)$ will be the amount at the end of the succeeding year. Hence, we have,

$$\begin{aligned} M_1 &= P(1+R), \\ M_2 &= M_1(1+R), \\ M_3 &= M_2(1+R), \\ &\vdots \qquad \vdots \\ M_n &= M_{n-1}\,(1+R). \end{aligned}$$

Multiplying all these equations together, we have,

$$M_1, M_2, \ldots M_n = P.M_1.\,M_2 \ldots M_{n-1}(1+R)^n,$$

Dividing by the factors common to each side,

$$M_n = P(1+R)^n \ldots (1.)$$

Or, substituting the value for R,

$$M_n = P\left(1 + \frac{r}{100}\right)^n \ldots (2.)$$

EXAMPLE. What will be the amount of £100 in 3 years at 5 per cent.

Here, $P = 100$, $r = 5$, and $n = 3$, therefore we have, by (2.), $M_3 = 100\left(1 + \frac{5}{100}\right)^3 = £115{\cdot}7625$.

ANNUITIES.

58. PROB. To find the amount, M, of an annuity, £A, payable yearly, for n number of years.

Here, the first payment will be in hand $n-1$ year, the second $n-2$ years, the third $n-3$ years, and so on; and the value of the annuity, therefore, will be found by adding together the amounts, at compound interest, of all these payments. Hence, we have, by Eq. (1.) Art. **57.**

$$\begin{array}{lll} \text{The amount of the 1st payment} & = A(1+R)^{n-1} \\ \ldots \quad \ldots \quad \text{2nd} \ldots & = A(1+R)^{n-2} \\ \ldots \quad \ldots \quad \text{3rd} \ldots & = A(1+R)^{n-3} \\ \quad\quad\quad \vdots & \quad \vdots \\ \text{The amount of the last payment} & = A. \end{array}$$

Hence we have, by adding all these amounts, and writing the terms in an inverse order,

Total amount, or $M = A + A(1+R) + A(1+R)^2 + \ldots + A(1+R)^{n-1}$,

$$\therefore M = A\left\{1 + (1+R) + (1+R)^2 + \ldots + (1+R)^{n-1}.\right\}$$

Taking the sum of the geometrical series (See Ex. 10., Art. **52.**),

$$M = A \times \frac{(1+R)^n - 1}{R} \ldots (1.)$$

This result gives the value of the annuity at the expiration of n years.

59. PROB. To find the *present* value of the annuity of Art. **58.**

Let P = the present value;

Then the amount of P, at compound interest, for n years, ought to produce M, the amount or worth of the annuity at the expiration of n years; but the amount of P for n years, (by Art. **57.**) $= P(1+R)^n$;

$$\therefore \text{ by (1.), Art. 58., } P(1+R)^n = A \times \frac{(1+R)^n - 1}{R}.$$

$$\therefore P = \frac{A}{R} \times \left\{1 - \frac{1}{(1+R)^n}\right\} \ldots (1.)$$

If the annuity arise from the rent of a freehold property; then the annuity is perpetual, and $\therefore$ n is infinitely large, and consequently $\frac{1}{(1+R)^n}$ becomes infinitely small, and the expression (1.), in this case simply becomes,

$$P = \frac{A}{R}\ ; \text{ or substituting } \frac{r}{100} \text{ for } R,$$

$$P = \frac{100A}{r} \ldots (2.)$$

This expression gives the purchase-money of an estate producing an annual rental of £A, when the value of money is expressed by r per cent.

If, for example, money is at present worth 4 per cent., then $P = \frac{100A}{4} = 25$ A, that is, the purchase-money of the estate ought to be 25 times the annual rent.

60. CUBIC EQUATIONS.

1. Given the cubic equation $x^3 + 6x = 2$, to find the value of x.

In order to bring equations of this kind to the form of a quadratic, take *the third part of the coefficient of x with its sign changed*, that is $\frac{1}{3}$ of $-6 = -2$, then substitute $z - \frac{2}{z}$ for x in the given equation, and the resulting equa-

tion, after reduction, will have the form of a quadratic: thus

$$x^3+6x=\left(z-\frac{2}{z}\right)^3+6\left(z-\frac{2}{z}\right), \text{ expanding (Art. 35. Ex. 9.}$$

$$=z^3-6z+\frac{12}{z}-\frac{8}{z^3}+6z-\frac{12}{z}$$

$$=z^3-\frac{8}{z^3}; \text{ but the proposed eq. is equal to } 2,$$

$$\therefore\ z^3-\frac{8}{z^3}=2,$$

multiplying each side by z^3,

$$z^6-8 \quad = 2z^3,$$

transposing, $z^6-2z^3=8$.

Now this equation has the form of a quadratic. (See Art. 42. Ex. 25.) Therefore by completing the square we have,

$$z^6-2z^3+1=9,$$

Taking the square root of each side,

$$z^3-1=3,$$

$$\therefore\ z^3=4, \text{ and } z=\sqrt[3]{4}=1{\cdot}587401.$$

But we have for the value of x in the original equation,

$$x=z-\frac{2}{z}=1{\cdot}587401-\frac{2}{1{\cdot}587401}={\cdot}32748.$$

Obs. By precisely the same method the following equations may be solved. The method, however, fails when the equation is such as to require us, in the course of the investigation, to take the square root of a minus quantity.

2.	$x^3+9x=6.$	*Ans.* $x={\cdot}637834.$
3.	$x^3+12x=12.$	... $x={\cdot}932441.$
4.	$x^3+6x=7.$	... $x=1.$
5.	$x^3+9x=-6.$	... $x=-{\cdot}637835.$
6.	$x^3+15x=20.$	... $x=1{\cdot}214042.$
7.	$x^3-3x=2.$	... $x=2.$
8.	$x^3-27x=-54.$	... $x=-6.$

61. When the given equation contains all the powers of x, then we must first remove the term containing x^2, before we proceed to apply the preceding method of solution.

1. Given $y^3-3y^2+12y=4$, to find the value of y.

Here we must first remove the second term. In order to do this, we take *the third part of the coefficient of y^2 with its sign changed*, that is $\frac{1}{3}$ of $3=1$, then substitute $x+1$ for y in the given equation, and the resulting equation, after reduction, will not contain the second term: thus,

$$y^3-3y^2+12y=(x+1)^3-3(x+1)^2+12(x+1)$$
$$=x^3+9x+10, \text{ by expansion and reduction.}$$
$$\therefore\ x^3+9x+10=4,$$
$$\text{and}\qquad x^3+9x=-6.$$

Then solving this equation by the preceding method, we find, $x=-\cdot637835$.

But $y=x+1=-\cdot637835+1=\cdot362165$ *Ans.*

2. $y^3+3y^2+24y=20$. *Ans.* $y=\cdot746375$.

62. Various particular methods are used for the solution of cubic and other equations. The following examples exhibit some of the most simple and elegant artifices.

1. Given $x^3-2x=4$, to find the value of x.

This equation may be solved by making each side a square quantity,

First multiplying each side of the equation by x,

$$x^4-2x^2=4x,$$

adding $4x^2$ to each side,

$$x^4+2x^2=4x^2+4x,$$

adding 1 to each side to complete the square,

$$x^4+2x^2+1=4x^2+4x+1,$$

taking the square root of each side,

$$x^2+1=2x+1,$$
$$\therefore\ x^2=2x,$$

then, dividing each side by x, we have, $x=2$.

2. $x^3+4x=16$. *Ans.* $x=2$.
3. $x^3-8x=3$. ... $x=3$.
4. $x^3-7x=-6$. ... $x=1$.

5. Given, $x^4+2x^3-x=2$, to find the value of x.

By adding and subtracting x^2, the equation becomes,

$$x^4+2x^3+x^2-(x^2+x)=2.$$
$$\therefore\ (x^2+x)^2-(x^2+x)=2.$$

We proceed now to solve this equation as a quadratic for the value of x^2+x. By completing the square, we have,

$$(x^2+x)^2-(x^2+x)+\tfrac{1}{4}=\tfrac{9}{4} \dots (1.)$$

taking the square root of each side,

$$x^2+x-\tfrac{1}{2}=\tfrac{3}{2},$$

solving this quadratic equation, we find,

$$x=1 \text{ or } -2.$$

By taking the minus value for the root of $\frac{9}{4}$ in Eq. (1.), two other values for x may be found, viz., $x=\frac{1}{2}(\pm\sqrt{5}-1.)$

6. $x^4-2x^3+x=30$. *Ans.* $x=3$ or -2.

7. $x^{4n}+2x^{3n}-x^n=2$. ... $x=1$ or $\sqrt[n]{-2}$.

8. Given $x^4+6x^3+6x+1=\dfrac{77x^2}{4}$, to find x.

Dividing every term by x^2,

$$x^2+\frac{1}{x^2}+6\left(x+\frac{1}{x}\right)=\frac{77}{4},$$

adding 2 to each side in order to render $x^2+\frac{1}{x^2}$ a square quantity,

$$\left(x+\frac{1}{x}\right)^2+6\left(x+\frac{1}{x}\right)=\frac{85}{4},$$

Solving this equation as a quadratic for $x+\dfrac{1}{x}$,

$$x+\frac{1}{x}=\frac{5}{2}.$$

Solving this quadratic, we find $x=2$ or $\frac{1}{2}$.

Equations of this kind are called *reciprocal equations.*

9. $x^4+2x^3+2x+1=\dfrac{142x^2}{9}$.

Ans. $x=3$, or $\frac{1}{3}$, or $\frac{1}{3}(\pm\sqrt{55}-8.)$

10. Given $(3x^2+13)^{\frac{1}{2}}-(3x^2+4)^{\frac{1}{2}}=1$, to find x.

By simply subtracting, we have,

$$(3x^2+13)-(3x^2+4)=9,$$

Dividing this equation by the given one (see Art. **34.**),

$$(3x^2+13)^{\frac{1}{2}}+(3x^2+4)^{\frac{1}{2}}=9,$$

adding this equation and the given one together,

$$2(3x^2\quad 13)^{\frac{1}{2}} = 10,$$

by dividing by 2, squaring each side, &c. we readily find $x = \pm 2$.

11. $(x^{\frac{1}{2}}+4)^{\frac{1}{2}}-(x^{\frac{1}{2}}-4)^{\frac{1}{2}} = 2$. *Ans.* $x = 25$.

The following equations may be solved by getting rid of a factor common to both sides of the equality.

12. Given $x^3-7x = -6$, to find the values of x.
Adding $6x$ to each side,

$$x^3-x = 6x-6; \text{ or } x(x^2-1) = 6(x-1) \ldots (1.),$$

dividing each side by $x-1$,

$$x(x+1) = 6,$$

solving this quadratic, we find $x = 2$, or -3.

In this example there is another root, which has been lost sight of by dividing each side of the equation by $x-1$; this factor, indeed, contains the lost root; for, by making $x-1 = 0$, we must obtain a value for x, which will obviously make Eq. (1.) a true equality, and consequently also the proposed equation. The other root will therefore be $x = 1$.

13. $x^3-19x = 30$. *Ans.* $x = 5$, or -2, or -3.

14. $x^3-76x = -240$. ... $x = 6$, or -10, or 4.

The best *general* method for finding the roots of equations, higher than the second order, is Horner's method of approximation. See Davies's edition of Hutton's Course, Thompson's Algebra, or Young's "Theory of Equations."

APPLICATION OF ALGEBRA TO MENSURATION.

1. Find the sides of a rectangle containing 135 square ft., and having its length and breadth in the ratio of 5 to 3.

Let $5x$ = the length in ft.,

Then $3x$ = the breadth in ft.,

$\therefore$ the area of the rectangle in sq. ft $= 5x \times 3x = 15x^2$.

But, by the question, the area is 135 sq. ft.

$\therefore\ 15x^2 = 135$; and $\therefore\ x = 3$ ft.

Hence the length = 5 times 3 ft. = 15 ft.; and the breadth $= 3\times 3 = 9$ ft.

2. Find the base of a triangle containing 240 sq. ft., and having the base and perpendicular in the ratio of 6 to 5.

Ans. 24 ft.

Here it will be observed, that the area of a triangle is found by multiplying the base by the perpendicular, and taking half the product.

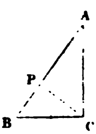
3. A plank 20 ft. long contains 30 sq. ft., what is its breadth? *Ans.* 18 in.

4. Find the side of a square containing 84 sq. yards.

Ans. 9·165.

5. The area of a triangle is 16 sq. ft., and the base is 4 ft.; required the perpendicular. *Ans.* 8 ft.

6. The hypothenuse BA of a right-angled triangle is 20, and the ratio of the base, BC, to the perpendicular, AC, is 3 to 4; required the base.



Let BC $= 3x$; then AC $= 4x$.

Now, in every right-angled triangle we have,

$$AB^2 = BC^2 + AC^2,$$

$$\therefore 20^2 = (3x)^2 + (4x)^2.$$

Solving this equation, we find, $x = 4$, and $\therefore$ BC $= 3\times 4 = 12$.

7. The hypothenuse of a right-angled triangle is 10, and the difference of the base and perpendicular is 2; required the sides. *Ans.* 6 and 8.

8. The diagonal of a square is 24 ft.; what is the side?

Ans. 16·97 ft.

9. The radius of a circle is 4 ft.; required the side of the inscribed square. *Ans.* 5·65 ft.

10. A rectangle contains 27 sq. ft., and the difference between the adjacent sides is 6 ft.; required the sides.

Let $x =$ the breadth; then the length $= x+6$; and the area $= x(x+6)$. But the area is 27 sq. ft.

$$\therefore x(x+6) = 27.$$

Solving this quadratic equation, we find $x = 3$ ft., and the length $= 3+6 = 9$ ft.

11. The sum of the adjacent sides of a rectangle is 7, and the area 10; required the sides. *Ans.* 5 and 2.

12. The area of a right-angled triangle $= 40$, and the sum of the base and perpendicular $= 18$; required the base. *Ans.* 10.

13. To find the side of a square whose area is equal to twice the sum of the sides. *Ans.* 8.

14. To find the side of a cube whose solid content is twice its surface. *Ans.* 12.

15. A person wants a cylindrical vessel 4 ft. deep to contain 20 c. ft. of water; required the diameter of the end.

Let $x =$ the diameter of the end in ft.,

then, the content of the cylinder in c. ft. $= x^2 \times \cdot 7854 \times 4$.

But the cylinder is to contain 20 c. ft.

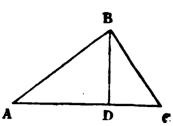
$$\therefore x^2 \times \cdot 7854 \times 4 = 20.$$

From this equation we find $x = 2\cdot 523$ ft.

16. A hollow cylinder, 10 ft. long, is to be cast out of 12 c. ft. of metal; what must be the exterior diameter, when the thickness is 2 inches? *Ans.* 29·5 inches.

17. To find the diameter of the end of a cylinder, which is a ft. long, when the surface of the two ends is equal to the convex surface. *Ans.* $2a$.

18. In the triangle ABC to find the segments of the base, AD and DC, when AC $= 12$, AB $= 8$, and BC $= 6$.



Let $x =$ AD, then $12-x =$ DC. Now, the right-angled triangles, ABD and DBC, will enable us to find two independent expressions for BD^2, thus,

$$BD^2 = 8^2 - x^2; \text{ and also } BD^2 = 6^2 - (12-x)^2.$$

$$\therefore 36 - (12-x)^2 = 64 - x^2,$$

$$\therefore 24x = 172; \text{ and } x = 7\tfrac{1}{6}, \therefore DC = 12 - 7\tfrac{1}{6} = 4\tfrac{5}{6}.$$

19. The segments of the base, AD and DC, are respec-

tively 12 and 9, and the sides AB and BC are in the ratio of 5 to 4; required the sides. See the last fig..

Ans. 13·228. and 10·583.

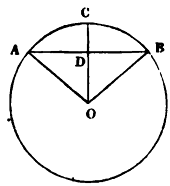
20. The base of a triangle = 5, the sum of the squares of the two other sides = 45, and the perpendicular on the base = 4; required the sides. *Ans.* 5 and $\sqrt{20}$.

21. Required the same as in the last example, when the difference of the squares of the two sides =15.

Ans. 5·656 and 4·123.

22. The diagonal of a rectangle is 50, and the perpendicular let fall upon the diagonal from the opposite angle is 24; required the sides. *Ans.* 30 and 40.

23. In the circular arch ACB, the span AB = 20, and the height DC = 6; required the radius OA.



Here, the right-angled triangle AOD will enable us to find an equation containing OA. Let x = OA or OC, then OD = OC−DC = $x-6$, and AD $=\frac{1}{2}$ AB = 10.

$$OA^2 = AD^2+OD^2;\ \therefore\ x^2 = 10^2+(x-6)^2.$$

Solving this equation, we find, $x = 11\frac{1}{3}$ ft.

24. To find the height DC of an arch, when the span AB = 18 ft., and the radius OA = 15 ft. *Ans.* 3 or 27 ft.

25. The radius OA = 4 ft., and the area of the sector AOBC = 12 sq. ft.; required the number of degrees in the arc ACB.

Let x = the number of degrees,

then, area of the whole circle = diam.$^2\times\cdot7854 = 8^2\times\cdot7854$.

Now, the whole circle is made up of 360 sectors of 1 degree each,

$$\therefore \text{ area sector of } 1^\circ = \frac{8^2\times\cdot7854}{360},$$

$$\therefore \quad \dots \quad \dots \quad x \text{ degrees} = \frac{8^2\times\cdot7854\times x}{360}.$$

By putting this expression equal to 12, the area given in

the question, we obtain an equation, from which we find $x = 85^\circ\ 56'\ 36''$.

26. The area of the segment ADBC of a circle containing $90^\circ = 8$ sq. ft.; required the radius.

Let $x =$ the radius AO.

As the arc ACB contains 90°, AOB will be a right-angled triangle.

$$\text{Area segt.} = \text{sector AOBC} - \text{triangle AOB},$$
$$= \frac{(2x)^2 \times \cdot 7854}{4} - \frac{x^2}{2}.$$

By putting this expression equal to 8, we obtain an equation, from which we find $x = 5 \cdot 29$.

27. The diameter of a circle is 10 ft.; it is required to find the sides of the inscribed rectangle whose area is 48 ft.

Ans. 6 and 8 ft.

28. Required the side of a square inscribed in a semicircle whose radius is 8. *Ans.* 7·15.

29. Let ABC be an equilateral triangle having its side $= 8$ ft.; required the area. See fig. to Prob. 18.

In order to find the area we must first find the perpendicular BD: for this purpose let $x =$ BD, then as AD $=$ DC $= \frac{1}{2}$ of $8 = 4$, we have from the right-angled triangle ABD,

$$x^2 + 4^2 = 8^2; \ \therefore x = \sqrt{48},$$

and, area triangle $= 4 \times \sqrt{48} = 27 \cdot 71$ ft.

30. The area of an equilateral triangle is 16 sq. yards; required the side. *Ans.* 6·075 yds.

31. The sum of the side and diagonal of a square $= 8$ ft. required the side. *Ans.* 3·313.

32. The sum of the three sides of a right-angled isosceles triangle $= 20$; required the equal sides. *Ans.* 5·85.

33. The difference between the base and perpendicular of an equilateral triangle is 4 ft.; required the side.

Ans. 29·856 or 2·143 ft.

34. The area of a right-angled triangle $= 24$, and the

three sides are in arithmetical progression; required the sides.

Let $x-y$, x, and $x+y$, be the three sides; where y is the common difference, then by the properties of the figure, and the data of the problem, we have the two following equations:

$$(x-y)^2+x^2=(x+y)^2 \text{ ...... } (1.)$$
$$x(x-y)=48 \text{ .... } (2.)$$

Reducing eq. (1), we find, $x=4y$ .... (3.)

Substituting this value for x in eq. (2.), we have,

$$4y \times 3y = 48, \text{ and } \therefore y=2.$$

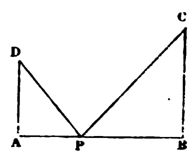
Putting this value for y in eq. (3), we have, $x=4\times 2=8$.

Hence the sides are, $8-2=6$, 8, and $8+2=10$.

35. Find the sides of a right-angled triangle, having its sides in geometrical progression, and area $=36$.

Ans. 7·52, 9·56, 12·17.

36. Two towers AD and BC, are 30 and 70 ft. high respectively, and their distance apart, AB, is 100 ft. At what point, P, will they appear to have the same elevation?



Let $x=\text{AP}$; then $100-x=\text{PB}$.

Now when the towers appear to have the same elevation, the angle DPA will be equal to the angle CPB, and consequently the triangles DAP and CBP will be equiangular and similar.

$$\therefore \text{AP} : \text{AD} :: \text{PB} : \text{BC},$$

that is, $x : 30 :: 100-x : 70$;

Therefore by Art. 50., $70x=30(100-x)$.

Solving this equation, we find, $x=30$ ft.

37. At what point, P, between the towers will $\text{DP}=\text{PC}$?

Ans. $\text{AP}=70$ ft.

38. Required the point P, where $\text{PC}^2-\text{PD}^2=400$ ft.

Ans. $\text{AP}=68$ ft.

39. The shadow of a stick, 4 ft. long, was 6 ft., at the time when the shadow of a tower was 90 ft.; required the height of the tower.

Ans. 60 ft.

40. The side of an equilateral triangle is 8; required the side of the inscribed square. *Ans.* 3·714.

41. A deal is 8 ft. long, 12 inches broad at the less end, and 18 inches at the greater; how much must be cut off from the less end to make 2 sq. feet? *Ans.* 1·888 ft.

42. The hypothenuse of a right-angled triangle is 50 ft., and the perpendicular, CP, let fall upon the hypothenuse is 24 ft.; required the sides. See fig. to Problem 6.

Let $x = BC$, and $y = CA$, then *double the area of the triangle* $= xy$; but we have also, *double the area of the triangle* $= 50 \times 24 = 1200$,

$$\therefore\ xy = 1200 \ \ldots\ (1.)$$
$$\text{and } x^2 + y^2 = 2500 \ \ldots\ (2.)$$

Solving these equations after the manner of Ex. 5., Art. 45.; we find $x = 40$, and $y = 30$.

43. The two sides of a right-angled triangle are 9 and 12; required the perpendicular on the hypothenuse. *Ans.* 7·2.

44. In a right-angled triangle, the two lines drawn from the acute angles bisecting the opposite sides are 4 and 6; required the sides. *Ans.* 2·732 and 5·842.

45. The sides of a right-angled triangle are 12 and 16; required the radius of the inscribed circle. *Ans.* 4.

46. A garden is 300 ft. long, and 54 ft. broad; what must be the breadth of a walk going round it, which shall contain 1400 sq. ft.?

Let $x =$ the breadth of the walk, then the sides of the enclosed rectangle will be $300 - 2x$, and $54 - 2x$, respectively.

$\therefore$ The area of the enclosed rectangle $= (300 - 2x)(54 - 2x)$. But from the question this area will be $300 \times 54 - 1400 = 14800$.

$$\therefore\ (300-2x)(54-2x) = 14800.$$

From this equation we find $x = 2$ ft.

47. The side of a square field is 180 yards; required the

breadth of a coach road going round it which shall contain 1424 sq. yards. *Ans.* 2 yards.

48. If the weight of a cubic foot of iron is 7200 oz., what must be the diameter of a ball weighing 24 lbs.?

Ans. 5·6 inches.

49. A solid inch of copper is made into a hollow sphere; what must be its diameter so that it may just swim in water, allowing that a cubic foot of copper weighs 8915 oz.?

Ans. 2·57 inches.

50. How many cubic inches of cork must be tied to a solid inch of copper, so as to make it swim in water, allowing that a cubic foot of cork weighs 240 oz.?

Ans. 10·41 c. in.

51. The perpendicular height of an equilateral triangle is 3 ft.; required the side. *Ans.* $\sqrt{12}$.

52. A circular walk, 3 ft. wide, is to be made round a fountain, what must be the radius of the outer circle so that the walk may contain 150 sq. ft.? *Ans.* 18·91 ft.

53. From what height above the earth will a person see one-third of its surface? *Ans.* The diameter of the earth.

54. To find the side of an equilateral triangle, the radius of the inscribed circle being 10. *Ans.* 34·64

55. In a right-angled triangle the sum of the two sides BC and AC = 35; and the sum of the hypothenuse AB, and perpendicular CP let fall upon it = 37; required the sides. See fig. to Problem 6.

Let $x = \text{BC}$, $y = \text{AC}$, $z = \text{AB}$, and $w = \text{CP}$, then we have, after the manner of Problem 42.,

$$x+y = 35 \ \ldots \ (1); \quad w+z = 37 \ \ldots \ (2.);$$
$$x^2+y^2 = w^2 \ \ldots \ (3); \quad xy = wz \ \ldots \ (4.).$$

Subtracting (3) from the square of (1.),

$$2xy = 35^2-w^2.$$

Multiplying (4) by 2, equating, and reducing,

$$w^2+2wz = 35^2.$$

Subtracting this equation from the square of (2.),

$$z^2 = 37^2-35^2; \ \therefore \ z = 12.$$

Hence we readily find $w = 25$, $x = 15$, and $y = 20$.

56. The difference of the legs of a right-angled triangle $= 10$, and the difference of the hypothenuse and the perpendicular from the right angle $= 26$; required the sides.

Ans. 30, 40, and 50.

57. A circle whose radius is 8 is inscribed in an isosceles triangle, and then another circle, whose radius is 4, is described, touching the former and the two equal sides of the triangle; required the perpendicular upon the base.

Ans. 32.

58. If a mountain h miles high can be seen at the distance of d miles; what must be the diameter of the earth?

Ans. $\frac{d^2 - h^2}{h}$.

THE END.

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